Joyce Seminar.

Cookin' with the Fixed Point Property

L'intelligenza cominci la sua opera: lungo il cammino non mancheranno certo i dolori che si assumeranno il compito di portarla a compimento. Quanto alla felicità essa non ha quasi che un'unica utilità: rendere possibile l'infelicità. Occorre che nella felicità si formino legami molto forti e dolci, di fiducia e di tenerezza, affinché la loro rottura ci susciti quella lacerazione così preziosa che si chiama infelicità. Se non fossimo stati felici, non foss'altro che a causa della speranza, le sventure sarebbero prive di crudeltà e di conseguenza resterebbero infruttuose.

Marcel Proust
Le temps retrouvé

Chapter 1

A first summary

1.0.1 A Brief History

Perhaps the most frequently cited fixed point theorem in analysis is the "Banach-Caccioppoli contraction mapping principle", which states that if (M, d) is a complete metric space and

$$f : M \longrightarrow M$$

is a contraction mapping, i.e.

$$\exists\ 0 < k < 1 \text{ such that } d(f(x), f(y)) \leq k \cdot d(x, y) \quad \forall x, y \in M,$$

then f has a unique fixed point in X (there is a unique $x_0 \in M$ such that $f(x_0) = x_0$).

This theorem has its origins in Euler and Cauchy's work [6] on the existence and uniqueness of a solution to the differential equation

$$\begin{cases} dy/dx = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

when f is a continuously differentiable function. In 1877, Lipschitz [9] simplified Cauchy's proof using what we now know as the "Lipschitz condition". In 1890 Picard [13] applied the method of iterations to ordinary equations as well as to a class of partial differential equations. The formulation of the theorem given above is due to Banach [2]. An interesting generalization of the Banach-Caccioppoli contraction principle was given by Ekeland [7].

The Lipschitz condition $k < 1$ is crucial even for the existence part of the result, but within more restrictive setting an amplified fixed point theorem exists for the case $k = 1$. Mappings which satisfy the condition for $k = 1$ are known as *non expansive*, and the theory of non expansive mappings is fundamentally different from that of contraction mappings. For example, even if a non expansive mapping f has a non empty set of fixed points $Fix(f)$, the Picard iterates may fail to converge. Also, $Fix(f)$ need not contain just one point.

Before we state the fixed point problem in Banach spaces, let us discuss the linear case, which is where the whole theory originated. Possibly the most important result in this case is the following

Theorem 1.1. *(Brouwder, [4] [5])*
For each $n \in \omega$, let $B_{\mathbb{R}^n}$ be the closed unit ball of $\mathbb{R}^n$. Then, any continuous mapping $f : B_{\mathbb{R}^n} \longrightarrow B_{\mathbb{R}^n}$ has a fixed point.

This result was previously known to Poincare [15] in an equivalent form. The underlying causes behind Brouwer 's theorem are the compactness and convexity of the unit ball of $\mathbb{R}^n$. Thus in [16, 17], Schauder extended Brouwer's theorem to obtain the same conclusion for any compact convex set in any linear topological space which is locally convex.

1.0.2 Normal structure and fixed point property

In this section, we shall indicate how the notions of strict convexity and uniform convexity come to play a role in the theory of fixed points of certain non-linear operators. Before to start, let us recall a geometrical notion

Definition 1.2. A Banach space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$ we have that $\|\frac{x+y}{2}\| \leq 1 - \delta$.

It is a classical result that every uniformly convex Banach space is reflexive.

Definition 1.3. A point x of a closed bounded convex C of a Banach space X is said to be *diametral* whenever

$$diamC = \sup\{\|y - x\| : \ y \in C\}.$$

Definition 1.4. (Brodskii and Milman, [3])
A bounded, closed, convex set K is said to have *normal structure* whenever given any closed bounded convex subset C of K containing more than one point there exists a non-diametral $x \in C$.

In other words, K has *normal structure* if every bounded convex non-void subset C of K with positive diameter

$$d = diamC = \sup\{\|x - y\| : \ x, y \in C\}$$

is contained in some ball centered in C with radius smaller than d.

A Banach space is said to have *normal structure* if any bounded, closed, convex of its subsets has normal structure.

Theorem 1.5. *Compact convex sets have normal structure.*

Proof. If K is a convex subset of the Banach space X and K does not have normal structure ($diamK > 0$) then for any $x_1 \in K$ there is $x_2 \in K$ such that $\|x_1 - x_2\| = diamK$. But $x_1, x_2 \in K$ implies that $\frac{x_1 - x_2}{2} \in K$. Thus there exists $x_3 \in K$ such that

$$\|x_3 - \frac{x_1 - x_2}{2}\| = diamK.$$

In this way we get a sequence $(x_n)_n$ of members of K for which

$$\|x_{n+1} - \frac{x_1 + \cdots + x_n}{n}\| = diamK.$$

But the

$$\begin{aligned} diamK = \|x_{n+1} - \frac{x_1 + \cdots + x_n}{n}\| &= \|\frac{x_{n+1} - x_1}{n} + \cdots + \frac{x_{n+1} - x_n}{n}\| \\ &\frac{1}{n}\sum_{i=1}^{n} \|x_{n+1} - x_i\| \\ &\leq diamK. \end{aligned}$$

Thus $\|x_{n+1} - x_i\| = diamK$ for each $i = 1, \ldots, n$. It follows that the sequence $(x_n)_n$ has no Cauchy subsequences, i.e., K is not compact. □

Similarly one can shows that

Theorem 1.6. *Closed bounded convex subsets of uniformly convex Banach spaces have normal structure.*

Definition 1.7. Let C be a subset of the Banach space X. A map $U : C \longrightarrow X$ is said to be *non expansive* whenever for $x, y \in C$

$$\|U(x) - U(y)\| \leq \|x - y\|$$

holds.

Theorem 1.8. *Let K be a weakly compact convex subset of a Banach space X. Suppose K possesses normal structure. Then each non-expansive $U : K \longrightarrow K$ has fixed point.*

Proof. Before to give the proof, we introduce some useful notion.

$r_x(K) = \sup\{\|x-y\| : \ y \in K\}$

$r(K) = \inf\{r_x(K) : \ x \in K\}$ (the radius of K)

$K_c = \{x \in K : \ r_x(K) = r(K)\}$

Let us note the following

(i) K_c is a non-empty closed convex subset of K.

In fact, consider $K_n(x) = \{y \in K : \ \|x-y\| \leq r(K) + \frac{1}{n}\}$. Then $\{K_n(x) : \ x \in K\}$ is a collection of weakly closed convex subsets of K possessing the finite intersection property. Thus

$$K_n = \bigcap_{x \in K} K_n(x)$$

is a non empty weakly closed convex set. Clearly $(K_n)_n$ is decreasing, thus $\bigcap_n K_n$ is a non empty weakly closed convex subset of K. Observe that $K_c = \bigcap_n K_n$.

(ii) $diam K_c < diam K$ (whenever $diam K > 0$).

In fact, as K has normal structure there exists $x \in K$ with $r_x(K) < diam K$. If $z, w \in K_c$ then $\|z-w\| \leq r_z(K) = r(K)$. Hence

$$diam K_c = \sup\{\|z-w\| : \ z, w \in K_c\} \leq r(K) \leq r_x(K) < diam K.$$

We are ready to prove the Theorem. Let $\mathcal{F}$ denote the collection of non empty closed convex subsets of K that are left invariant by U. Ordering $\mathcal{F}$ by inclusion and applying Zorn's lemma we get a minimal element F of $\mathcal{F}$ (Zorn's lemma is applicable due to the weak compactness of K). We will show that F is a singleton. Let $x \in F_c$. Then $\|U(x)-U(y)\| \leq \|x-y\| \leq r(F)$, for all $y \in F$. Thus, $U(F)$ is contained in the ball centered at $U(x)$ with radius $r(F)$. But $U(F \cap Ball(U(x), r(F)))$ is contained in $F \cap Ball(U(x), r(F))$. Thus by F's minimality we must have

$$F \subseteq ball(U(x), r(F)).$$

Since $U(x) \in F$, we must have $U(x) \in F_c$, i.e., $U(F_c) \subseteq F_c$. By the observation (i) F_c is a non empty closed convex subset of K. Therefore F_c is in $\mathcal{F}$. If $diam F > 0$, (ii) yields $diam F_c < diam F$, so $F_c \subsetneq F$. This contradict the minimality of F. It follows that $diam F = 0$, i.e., F is a singleton. □

Corollary 1.9. *If C is a non empty closed bounded convex subset of a uniformly convex Banach space, then every non-expansive $U : C \longrightarrow C$ has a fixed point.*

Proof. Since the space is uniformly convex, C has to have normal structure. Since every uniformly convex space is reflexive, C has to be weakly compact. An appeal to the previous theorem finishes the proof. □

Definition 1.10. A Banach space X is said to be *strictly convex* whenever S_X (the unit sphere of X) contains no non-trivial line segment, i.e., each point of S_X is an extreme point of B_X.

Theorem 1.11. *The fixed points of a non-expansive map $U : C \longrightarrow X$, where C is closed convex subset of the strictly convex space X, constitute a closed convex subset of X.*

Proof. Denote by $Fix(U)$ the set of all fixed points of U. $Fix(U)$ is clearly closed. Let us show that $Fix(U)$ is also convex. Indeed, let $x_1, x_2 \in Fix(U)$, $0 < \lambda < 1$ and consider $x = \lambda x_1 + (1-\lambda)x_2$. Then

$$\begin{aligned}
\|x_1 - x_2\| &\leq \|x_1 - U(x)\| + \|U(x) - x_2\| \\
&= \|U(x_1) - U(x)\| + \|U(x) + U(x_2)\| \\
&\leq \|x_1 - x\| + \|x - x_2\| \\
&= \|x_1 - \lambda x_1 - (1-\lambda)x_2\| + \|\lambda x_1 + (1-\lambda)x_2 - x_2\| \\
&= (1-\lambda)\|x_1 - x_2\| + \lambda\|x_1 - x_2\| \\
&= \|x_1 - x_2\|.
\end{aligned}$$

Thus

$$\|x_1 - x_2\| = \|x_1 - U(x)\| + \|U(x) - x_2\| \quad \text{and} \quad \|x_1 - x\| = \|x_1 - U(x)\|.$$

By the strict convexity of X, the first of these conclusion means that $U(x)$ is on the line segment connecting x_1 and x_2; the second yields $U(x) = x$. Thus $x \in Fix(U)$, and then $Fix(U)$ is convex. □

Theorem 1.12. *Let C be a weakly compact convex subset of a strictly convex Banach space X. Suppose C possesses normal structure. Let $U_\lambda : C \longrightarrow C$ ($\lambda \in \Lambda$) be a family of commuting non-expansive maps. Then U_λ's possess a common fixed point.*

Proof. For each $\lambda \in \Lambda$, by Theorem 1.8 and 1.11 we have that $Fix(U_\lambda)$ is a non empty weakly closed convex of C. By the weak compactness of C if we show that $\{Fix(U_\lambda)\}_\lambda$ possesses the finite intersection property then we will be done. To this end, note that if $x \in Fix(U_\lambda)$ then

$$U_\lambda U_\mu(x) = U_\mu U_\lambda(x) = U_\mu(x).$$

Thus $U_\mu(Fix(U_\lambda)) \subseteq Fix(U_\lambda)$. By Theorem 1.8 it follows that

$$Fix(U_\lambda) \cap Fix(U_\mu) \neq \emptyset.$$

Inductively, if $\lambda_1, \ldots, \lambda_n \in \Lambda$ and we consider $F = Fix(U_{\lambda_1}) \cap \ldots \cap Fix(U_{\lambda_n})$, then

$$U_{\lambda_{n+1}} : F \longrightarrow F$$

is non-expansive on the set F satisfying the hypotheses of Theorem 1.8, thus $U_{\lambda_{n+1}}$ has a fixed point in F, i.e.,

$$\bigcap_{i=1}^{n+1} Fix(U_{\lambda_i}) \neq \emptyset.$$

□

Definition 1.13. A bounded, closed, convex set K of a Banach space X is said to have the *fixed point property* (f.p.p.) if every non-expansive mapping U taking K to itself has non empty fixed point set ($Fix(U) \neq \emptyset$).

A Banach space X is said to have the *fixed point property* if every of its bounded, closed, convex subset has the f.p.p..

A Banach space X is said to have the *weak fixed point property* if every of its weakly compact convex subset has the f.p.p..

From what we have said above, any uniformly convex space has f.p.p., and any Banach space with normal structure has the weak fixed point property. For reflexive Banach space or even for super-reflexive Banach space the question is still to day open. It was conjectured for some period that any Banach space has the weak fixed point property. That was negatively solved by D. Alspace in 1980 (see [1]).

Example 1.14. Let $X = L_1[0,1]$ and let

$$K = \{f \in L_1[0,1] : \ 0 \leq f \leq 2, \ \|f\|_{L_1[0,1]} = 1\}.$$

It is easy to see that K is weakly closed, convex subset of the order interval $\{f : \ 0 \leq f \leq 2\}$, and thus K is weakly compact (because order intervals of $L_1[0,1]$ are weakly compacts). Let us define the map

$$T : K \longrightarrow K$$

given by

$$T(f)(t) = \begin{cases} 2f(2t) \wedge 2, & 0 \le t \le \frac{1}{2} \\ 2[f(2t-1) - 2, & \frac{1}{2} < t \le 1. \end{cases}$$

It is easy to check that actually T is an isometry on K.

Suppose that T has fixed point, i.e., there exists $g \in K$ such that $T(g) = g$. First note that necessarily $g = 2\chi_A$ for some measurable set A with measure $\frac{1}{2}$.

Indeed,

$$\begin{aligned} \{t:\ g(t) = 2\} &= \{t:\ T(g)(t) = 2\} \\ &= \{\frac{t}{2}:\ g(t) = 2\} + \{\frac{1+t}{2}:\ g(t) = 2\} + \{\frac{t}{2}:\ 1 \le g(t) < 2\} \end{aligned}$$

where + denotes the disjoint union.

Because the measure of $\{\frac{t}{2}:\ g(t) = 2\} + \{\frac{1+t}{2}:\ g(t) = 2\}$ is equal to the measure of $\{t:\ g(t) = 2\}$, it follows that the measure of $\{\frac{t}{2}:\ 1 \le g(t) < 2\}$ is zero. An iteration of this argument shows that

$$\{t:\ 0 < g(t) < 2\} = \bigcup_{n=0}^{\infty} \{t:\ 2^{-n} \le g(t) < 2^{-n+1}\}$$

has measure zero, as well.

Next observe that for $g = 2\chi_A$

$$\{t:\ T^n(g)(t) = 2\} = \sum_{\varepsilon_i \in \{0,1\}} \left\{ \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} + \cdots + \frac{\varepsilon_n}{2^n} + \frac{t}{2^n}:\ t \in A \right\}$$

for all n. We have this for $n = 1$ above, and for induction can be proved in general.

Since g is fixed we have $A = \{t:\ T^n(g)(t) = 2\}$ for all $n \in \omega$ and thus the intersection of A with any interval with dyadic end points has measure exactly half the measure of the interval. But no such measurable set exists in $[0, 1]$. This contradiction shows that T has no fixed point.

In the sequel, we are going to investigate the following two questions

Question 1.15. (i) Does any reflexive Banach space the (weak) Fixed Point Property?

(ii) Does any Banach space isomorphic to ℓ_2 the (weak) Fixed Point Property?

Before to go on, a stronger question could be if any Banach space isomorphic to ℓ_2 has normal structure. Since normal structure implies weak fixed point property, one may ask if the notions are equivalents. Those two sub-questions are solved in 1976 by Karlovitz [10].

1.0.3 Karlovitz's construction

Let X_J be the space ℓ_2 renormed according to

$$\|x\|_J = \max\{\|x\|_{\ell_\infty}, \frac{1}{\sqrt{2}}\|x\|_{\ell_2}\}.$$

This space was first originated by R.C. James. Of course, X_J is isomorphic to ℓ_2. Next result says that X_J is an example of space isomorphic to ℓ_2 which fails normal structure but still with the (weak) fixed point property.

Before to go on, let us recall some basic facts about non-expansive mappings.

Let$K$ be a non empty, bounded, closed, convex subset of a Banach space X. Let $T : K \longrightarrow K$ be a non-expansive mapping. Fix $n \in \omega$ and $z \in K$, and consider the mapping

$$T_n : K \longrightarrow K$$

defined by

$$T_n(x) = \frac{1}{n}z + (1 - \frac{1}{n})T(x)$$

for all $x \in K$. T_n is clearly a contraction mapping, and therefore has a unique fixed point $x_n \in K$. Then we have

$$x_n - T(x_n) = \frac{z - T(x_n)}{n} \to 0 \text{ as } n \to \infty$$

and by nonexpansiveness of T, for all $n \in \omega$

$$\begin{aligned}\|x_n - x_{n+1}\| &= \left\| \frac{1}{n(n+1)}(z - T(x_n)) + (1 - \frac{1}{n+1})(T(x_n) - T(x_{n+1})) \right\| \\ &\leq \frac{1}{n(n+1)}\|z - T(x_n)\| + (1 - \frac{1}{n+1})\|x_n - x_{n+1}\|\end{aligned}$$

and so

$$\|x_n - x_{n+1}\| \leq \frac{\|z - T(x_n)\|}{n} \to 0 \text{ as } n \to \infty$$

Definition 1.16. A sequence $(x_n)_n$ satisfying $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$ is called an *approximate fixed point sequence*, in short an a.f.p.s.

Let us suppose K weakly compact convex subset of X. Set

$$\mathcal{F} = \{C \subseteq K : C \text{ is non empty, closed convex, and invariant under } T \text{, i.e., } TC \subseteq C\}.$$

Clearly $\mathcal{F}$ is a non empty family, since $K \in \mathcal{F}$. It is easy to see that any decreasing chain of elements in $\mathcal{F}$ has a non empty intersection, because K is weakly compact, which belongs to $\mathcal{F}$. Therefore, one can use Zorn's lemma to demonstrate the existence of minimal elements of $\mathcal{F}$.

Definition 1.17. A convex set C is said to be *minimal* for T if C is a minimal element of $\mathcal{F}$.

Lemma 1.18. *Let K be minimal for T. Then*

$$\overline{conv}(TK) = K.$$

Proof. Let $K_0 = \overline{conv}(TK)$; clearly K_0 is non empty, closed, convex subset of K (since $TK \subseteq K$). Hence $TK_0 \subseteq TK \subseteq K_0$, so K_0 is invariant under T. Therefore $K_0 \in \mathcal{F}$ and since K is minimal we get $K_0 = K$. □

Lemma 1.19. *Let $\alpha : K \longrightarrow \mathbb{R}_+$ be a lower semi-continuous convex function. Assume that*

$$\alpha(Tx) \leq \alpha(x) \quad \textit{for all } x \in K.$$

Then α is a constant function.

Proof. Let $x_0 \in K$ be fixed. Define

$$K_0 = \{x \in K : \ \alpha(x) \leq \alpha(x_0)\}.$$

Then K_0 is non empty, closed, convex subset of K, since α is a lower semi-continuous convex function. Our assumption on α implies that K_0 is invariant under T, and since $x_0 \in K_0$, we deduce (by minimality of K) that $K_0 = K$. Therefore $\alpha(x) \leq \alpha(x_0)$ for all $x \in K$. But since x_0 was arbitrary, this complete the proof. □

Lemma 1.20. *The minimal set K is diametral, i.e.,*

$$\sup_{y \in K} \|x - y\| = diamK \quad \textit{for all } x \in K.$$

Proof. Set $\alpha(x) = \sup\{\|x - y\| : \ y \in K\}$. Then α is a continuous convex function. If $x \in K$, then $K \subseteq ball(x, \alpha(x))$; since T is non-expansive we deduce that $TK \subseteq ball(T(x), \alpha(x))$. By Lemma 1.18, $K = \overline{conv}TK \subseteq ball(T(x), \alpha(x))$. This obviously implies that $\alpha(Tx) \leq \alpha(x)$. By Lemma 1.19 α is constant. Say $\alpha(x) = \alpha$ for all $x \in K$. Since $\sup\{\|x - y\| : \ x, y \in K\} = diamK$ it follows that $\alpha = diamK$. □

Lemma 1.21. *Let K be a minimal set for T set for T. Then for any a.f.p.s. $(x_n)_n$ in K, we have*

$$\lim_{n \to \infty} \|x_n - x\| = diamK \quad \textit{for all } x \in K.$$

Proof. Set $\alpha(x) = \lim_{\mathcal{U}} \|x_n - x\|$, where $\mathcal{U}$ is an ultrafilter on ω. The function α is well founded because $(x_n)_n$ is bounded, and α is clearly continuous and convex. Since $(x_n)_n$ is an a.f.p.s. for T it follows that $\alpha(T(x)) \leq \alpha(x)$ for any $x \in K$. Therefore α satisfies the conditions of Lemma 1.19, so α must be a constant function, say $\alpha(x) = \alpha$. Using the weak compactness of K, we can deduce that the weak limit of $(x_n)_n$ over $\mathcal{U}$ exists in K. Put $z = weak-lim_{\mathcal{U}} x_n$. Since the norm is weak lower semi-continuous, we obtain

$$\|z - x\| \leq \lim_{\mathcal{U}} \|x_n - x\| = \alpha$$

for any $x \in K$. By Lemma 1.20 we have that $\alpha = diamK$. Since $(\|x_n - x\|)_n$ has a unique cluster point, it is convergent. This complete the proof. □

Theorem 1.22.

(i) E_J *does not have normal structure*

(ii) E_J *has (weak) fixed point property.*

Proof. (i) Let

$$K = \{x \in X_J : \ \|x\|_{\ell_2} \leq 1, \ x(i) \geq 0 \text{ for all } i \in \omega\}.$$

It is easily seen that K is bounded, closed and convex, to consist of more than one point and to have the property that

$$sup\{\|y - x\|_J : \ y \in K\} = \sqrt{2} = diam_{\|\cdot\|_J} K \text{ for all } x \in K.$$

(ii) Let C be a non empty weakly compact, convex subset of X_J and

$$T : C \longrightarrow C$$

be a non expansive map.

Given $x \in X_J$ we represent its component by $x(j)$, $j = 1, 2, \ldots$. Since X_J is a renorming of ℓ_2 there exists a component $x(j)$ so that $\|x\|_\infty = |x(j)|$.

Let $C_0 \subseteq C$ be a minimal invariant set for T. We propose to show that C_0 is a single point. By invariance this is then a fixed point of T.

We proceed by contradiction. Suppose that C_0 consists of more than one point. We may assume without loss of generality that $0 \in C_0$ and we let $diamC_0 = r > 0$. For each $0 < s < 1$ we define $T_s = (1 - s)T$. Clearly $T_s : C_0 \longrightarrow C_0$ and it is a strict contraction. Hence by the Banach-Caccioppoli contraction principle there exists a unique $x_s \in C_0$ so that $Tx_s = x_s$. Thus

$$Tx_s = \frac{x_s}{1 - s} \quad 0 < s < 1.$$

By minimality of C_0, $x_s \neq 0$. The desired contradiction results from a study of the points x_s. Several propositions are needed. □

Proposition 1.23. *For each* $x \in C_0$, $\lim_{s\to 0} \|x - x_s\|_\infty = r$.

Proof. By contradiction. Suppose that for some $x \neq 0 \in C_0$ and sequence $\{x_{s_n}\}_n$ with $s_n \to 0$, denoted simply by $\{x_n\}_n$, $\|x - x_n\|_\infty \leq r - \delta$, $\delta > 0$. Since $\|0 - x_n\|_\infty \leq diamC_0 = r$ it follows that $\|\frac{x}{2} - x_n\|_\infty \leq r - \frac{\delta}{2}$ for all n.

By the uniform convexity of $\|\cdot\|_2$, it follows form

$$\frac{1}{\sqrt{2}}\|x - x_n\|_2, \quad \frac{1}{\sqrt{2}}\|0 - x_n\|_2 \leq diamC_0 = r$$

that

$$\frac{1}{\sqrt{2}}\|\frac{x}{2} - x_n\|_2 \leq r - \tau$$

for some $\tau > 0$. Hence,

$$\|\frac{x}{2} - x_n\|_J \leq r - min\{\tau, \frac{\delta}{2}\}$$

for all n, which contradicts Lemma 1.21, because

$$\|T(x_n) - x_n\|_J = \frac{s_n}{1 - s_n}\|x_n\|_J \to 0.$$

□

Proposition 1.24. *For each* $0 < s < 1$, $\lim_{t\to s} \|x_t - x_s\|_J = 0$.

Proof. We denote x_s by x and x_t by y. Suppose that $\|x - y\| = \|x - y\|_\infty = |x(k) - y(k)|$. By nonexpansiveness

$$\left|\frac{x(k)}{1 - s} - \frac{y(k)}{1 - t}\right| \leq |x(k) - y(k)|. \tag{1.1}$$

For $s \neq t$ it follows that $signx(k) = signy(k) = \sigma = \pm 1$. Suppose that $0 < t < s$. If $\sigma x(k) > \sigma y(k)$ then

$$\frac{\sigma x(k)}{1 - s} - \frac{\sigma y(k)}{1 - t} > \frac{\sigma x(k) - \sigma y(k)}{1 - t},$$

contradicting (1.1). Hence $\sigma y(k) \geq \sigma x(k)$. If

$$\frac{\sigma y(k)}{1 - t} - \frac{\sigma y(k)}{1 - s} \geq 0$$

then, by (1.1),

$$\frac{(1 - s)t\sigma y(k)}{(1 - t)s} \leq \sigma x(k) \leq \sigma y(k)$$

If $(1-s)^{-1}\sigma x(k) - (1-t)^{-1}\sigma y(k) \geq 0$ then, directly

$$\frac{(1-s)\sigma y(k)}{1-t} \leq \sigma x(k) \leq \sigma y(k).$$

If $s < t < 1$, analogous inequalities are derived. It follows that

$$|x(k) - y(k)| \leq \begin{cases} (s-t)s^{-1}(1-t)^{-1}|y(k)|, & 0 < t < s, \\ (t-s)t^{-1}(1-s)^{-1}|x(k)|, & s < t < 1. \end{cases} \tag{1.2}$$

Hence if $\|x_t - x_s\|_J = \|x_t - x_s\|_\infty$ and $s/2 < t < (1+s)/2$,

$$\|x_t - x_s\|_J \leq A \cdot |s-t|, \quad \text{for some } A = A(s) > 0. \tag{1.3}$$

Now suppose that $\|x - y\|_J = \frac{1}{\sqrt{2}}\|x - y\|_2$. By nonexpansiveness

$$\left\| \frac{x}{1-s} - \frac{y}{1-t} \right\|_2 \leq \|x - y\|_2.$$

We divide the positive integers according to:

$$I_1 = \{i : \left| \frac{x(i)}{1-s} - \frac{y(i)}{1-t} \right| \leq |x(i) - y(i)|\}$$

and

$$I_1 = \{i : \left| \frac{x(i)}{1-s} - \frac{y(i)}{1-t} \right| > |x(i) - y(i)|\}.$$

Then

$$\sum_{I_2} [((1-s)^{-1}x(i) - (1-t)^{-1}y(i))^2 - (x(i) - y(i))^2] \leq \sum_{I_1} |x(i) - y(i)|^2. \tag{1.4}$$

By definition, (1.1) holds for $k \in I_1$. We can deduce, as above, that (1.2) holds. Whence, for $s/2 < t < (1+s)/2$,

$$\sum_{I_1} (x(i) - y(i))^2 \leq B(s-t)^2 \quad \text{for some } B = B(s) > 0. \tag{1.5}$$

We note the identity:

$$(1-t)^{-1}y(i) - (1-s)^{-1}x(i) = (1-s)^{-1}(y(i) - x(i) - \gamma(s,t)y(i)),$$

where $\gamma(s,t) = (s-t)(1-t)^{-1}$. Substitution into (1.4) yields

$$\sum_{I_2} [\frac{(y(i) - x(i) - \gamma(s,t)y(i))^2}{(1-s)^2} - (x(i) - y(i))^2] \leq \sum_{I_1} (x(i) - y(i))^2.$$

By Schwartz inequality and some simple manipulation

$$\sum_{I_2}(x(i)-y(i))^2 \le \frac{(1-s)^2}{s}\sum_{I_1}(x(i)-y(i))^2 + \frac{2\gamma(s,t)}{s}\|y\|_2\|x-y\|_2.$$

Combining this with (1.5) we find that if $\|x_s - x_t\|_J = \frac{1}{\sqrt{2}}\|x_s - x_t\|_2$ and $s/2 < t < (1+s)/2$ then

$$\|x_s - x_t\|_J \le K(s-t)^{1/2} \quad \text{for some } K = K(s) > 0. \tag{1.6}$$

The proposition now follows form (1.3) and (1.6). $\square$

For each positive integer i and $\varepsilon > 0$ we introduce the notation:

$$A^\varepsilon(i) = \{s:\ 0 < s < 1,\ |x_s(i)| \ge r - \varepsilon\} \text{ and } \alpha^\varepsilon(i) = \inf A^\varepsilon(i).$$

Proposition 1.25. *For each positive integer i and $0 < \varepsilon \le r/4$, there exists $0 < s_1 < 1$ with the property that for each $0 < s \le s_1$, there exists a positive integer $k(s)$ such that $k(s) \ne i$ and $|x_s(k(s))| \ge r - \varepsilon$.*

Proof. If $A^\varepsilon(i) = \emptyset$ this follows from Proposition 1.23 with $x = 0$. Otherwise choose $s_0 \in A^\varepsilon(i)$. Let $\varepsilon_1 = \min\{s_0(1-s_0)^{-1}(r-\varepsilon),\ \varepsilon/2,\ rs_0/2,\ s_0(1-s_0)^{-1}\varepsilon/2\}$. By Proposition 1.23 choose s_1 so that

$$\|x_{s_0} - x_s\|_\infty \ge r - \varepsilon_1 \text{ for all } 0 < s \le s_1.$$

Choose $0 < s \le s_1$ Suppose that $sign x_{s_0}(i) = sign x_s(i)$ or $x_s(i) = 0$. Then from $3r/4 \le |x_{s_0}(i)| \le r(1-s_0)$ we deduce that

$$|x_{s_0}(i) - x_s(i)| \le r - rs_0, \quad r/4 < r - \varepsilon_1.$$

If $sign x_{s_0}(i) = -sign x_s(i)$ then

$$\begin{aligned} r \ge \|Tx_{s_0} - Tx_s\| &\ge |\frac{x_{s_0}(i)}{1-s_0} - \frac{x_s(i)}{1-s}| \\ &> \frac{s_0}{1-s_0}|x_{s_0}(i)| \\ &\ge |x_{s_0}(i) - x_s(i)| + \frac{s_0}{1-s_0}(r-\varepsilon) \\ &\ge |x_{s_0}(i) - x_s(i)| + \varepsilon_1, \end{aligned}$$

and hence $|x_{s_0}(i) - x_s(i)| < r - \varepsilon_1$. Thus there exists a positive integer $j \ne i$ so that $\|x_{s_0} - x_s\|_\infty = |x_{s_0}(j) - x_s(j)| \ge r - \varepsilon_1$. We assert that $k(s) = j$ satisfies the proposition.

If $signx_{s_0}(j) = signx_s(j)$ then $r-\varepsilon_1 \leq |x_{s_0}(j)-x_s(j)| < \max\{|x_{s_0}(j)|; |x_s(j)|\}$. Since $|x_{s_0}(j)| \leq r(1-s_0) < r-\varepsilon_1$ it follows that

$$|x_s(j)| > r-\varepsilon_1,$$

as wished.

If $signx_{s_0}(j) \neq signx_s(j)$, then

$$\begin{aligned} r &\geq \|Tx_{s_0} - Tx_s\| \\ &\geq |\frac{x_{s_0}(j)}{1-s_0} - \frac{x_s(j)}{1-s}| \\ &= \frac{|x_{s_0}(j)|}{1-s_0} + \frac{|x_s(j)|}{1-s} \\ &\geq |x_{s_0}(j)| + \frac{s_0}{1-s_0}|x_{s_0}(j)| \\ &= |x_{s_0}(j) - x_s(j)| + \frac{s_0}{1-s_0}|x_{s_0}(j)| \\ &\geq r - \varepsilon_1 + \frac{s_0}{1-s_0}|x_{s_0}(j)|. \end{aligned}$$

Hence $|x_{s_0}(j)| \leq s_0^{-1}(1-s_0)\varepsilon_1$. So $|x_{s_0}(j) - x_s(j)| \geq r-\varepsilon_1$ implies

$$r - \varepsilon_1 - s_0^{-1}(1-s_0)\varepsilon_1 \geq r - \varepsilon_1 - \varepsilon/2 \geq r - \varepsilon,$$

as desired. Since $s \geq s_1$ was arbitrarily chosen this finishes the proof. □

Proposition 1.26. *Suppose* $\alpha^\varepsilon(i) = \alpha^\delta(j) = 0$ *for some* ε, δ *with* $\varepsilon > 0$ *and* $\delta \leq r/64$. *Then* $i = j$.

Proof. For i and ε we choose s_1 according to Proposition 1.25. Thus if $s \in A^\varepsilon(i)$ and $s \leq s_1$, then $|x_s(m)| \geq r - \varepsilon$ for $m = k(s) \neq i$.

Since $k(s) \neq i$ and $\|x_s\|_2^2 \leq 2r^2$ we readily find that $|x_s(m)| \leq r/4$ for $m \neq i, k(s)$. By Proposition 1.23 we choose $s_2, s_3 \in A^\varepsilon(i)$, $s_2, s_3 < s_1$ so that

$$\|x_{s_p} - x_{s_q}\|_\infty \geq r - \varepsilon \quad \text{for } p \neq q, \ (p,q = 1,2,3).$$

Now suppose that $k(s_1) = k(s_2)$. Then

$$|x_{s_1}(m) - x_{s_2}(m)| \leq |x_{s_1}(m)| + |x_{s_2}(m)| \leq r/2 < r - \varepsilon$$

for $m \neq i, k(s_1)$. Moreover, $signx_{s_1}(i) = signx_{s_2}(i)$; otherwise

$$\|x_{s_1} - x_{s_2}\|_\infty \geq |x_{s_1}(i)| + |x_{s_2}(i)| \geq 2r - 2\varepsilon > r.$$

Thus $|x_{s_1}(i) - x_{s_2}(i)| \leq r - (r - \varepsilon) = \varepsilon$.

By the same argument $|x_{s_1}(k(s_1)) - x_{s_2}(k(s_2))| \leq \varepsilon$. Thus $|x_{s_1}(i) - x_{s_2}(i)| < r - \varepsilon$ for all positive integer i, which is a contradiction.

Hence $k(s_1) \neq k(s_2)$. Similarly $k(s_3) \neq k(s_1), k(s_2)$. Now if $i \neq j$ we repeat the argument and find $t_1, t_2, t_3 \in A^\delta(j)$ so that $|x_{t_p}(j)|, |x_{t_p}(k(t_p))| \geq r - \delta$, $p = 1, 2, 3$ and so that $j, k(t_1), k(t_2)$ and $k(t_3)$ are disjoint. Thus we can find s_p and t_p so that $\{i, k(s_q)\} \cap \{j, k(t_p)\} = \emptyset$.

Then from

$$|x_{s_q}(i)|,\ |x_{s_q}(k(s_q))| \geq r - \varepsilon, \quad |x_{t_p}(j)|,\ |x_{t_p}(k(t_p))| \geq r - \delta$$

and

$$\|x_{t_p}\|_2,\ \|x_{s_q}\|_2 \leq \sqrt{2}r$$

it follows that

$$\frac{1}{\sqrt{2}}\|x_{s_q} - x_{t_p}\|_2 > r$$

which contradicts $x_{s_q},\ x_{t_p} \in C_0$. Hence $i = j$. □

Let us complete the proof of Theorem 1.22.

Let $\varepsilon = r/128$. If there exists a positive integer i so that $\alpha^\varepsilon(i) = 0$, let $i_0 = i$.

Otherwise $\alpha^\varepsilon(i) > 0$ for each i and we let $i_0 = 1$. Apply Proposition 1.25 to find $s_1 = s_1(i_0, \varepsilon)$. In the sequel the positive integer $k(\cdot)$ will be those given by the Proposition for this s_1. Denote by $k(s_1)$ by k_1.

Let $s_2 = \alpha^\varepsilon(k_1)$. If $\alpha^\varepsilon(i_0) = 0$ it follows from $i_0 \neq k_1$ and Proposition 1.26 that $s_2 > 0$; otherwise $s_2 > 0$ by hypothesis. By Proposition 1.24 $\|x_{s_2-\mu} - x_{s_2}\| \to 0$ as $\mu \to 0$. Hence we can choose $\mu > 0$ so that $r - 2\varepsilon \leq |x_{s_2-\mu}(k_1)|$.

Since $s_2 - \mu < s_2 < s_1$, $k(s_2 - \mu)$ is well defined. Since $s_2 - \mu < \alpha^\varepsilon(k_1)$, $|x_{s_2-\mu}(k_1)| < r - \varepsilon$; hence $k(s_2 - \mu) \neq k_1$.

Denote $x_{s_2-\mu}$ by y and $k(s_2 - \mu)$ by k_2.

Thus $|y(k_1)|,\ |y(k_2)| \geq r - 2\varepsilon$. Since $k_1, k_2 \neq i_0$, reasoning as above, $\alpha^\varepsilon(k_1), \alpha^\varepsilon(k_2) > 0$. Hence we can choose $0 < s_3 < \alpha^\varepsilon(k_1), \alpha^\varepsilon(k_2)$. Then

$$|x_{s_3}(k_1)|,\ |x_{s_3}(k_2)| < r - \varepsilon,$$

and hence $k_3 = k(s_3) \neq k_1, k_2$. repeating the argument we find $z = x_{s_3-\eta}$, $\eta > 0$, and $k_4 = k(s_3 - \eta) \neq k_3$ so that

$$|z(k_3)| \geq r - 2\varepsilon \quad \text{and} \quad |z(k_4)| \geq r - 2\varepsilon.$$

Moreover, $s_3 - \eta < \alpha^\varepsilon(k_1),\ \alpha^\varepsilon(k_2)$, hence $k_4 \neq k_1, k_2$. Thus k_1, k_2, k_3 and k_4 are disjoint. Hence from $\|y\|,\ \|z\| \leq r$ and $|y(k_1)|,\ |y(k_2)|,\ |z(k_3)|,\ |z(k_4)| \geq r - 2\varepsilon$ we readily get

$$\frac{1}{\sqrt{2}}\|y - z\|_2 > r$$

which contradicts $y, z \in C_0$. This contradiction proves that C_0 cannot consist of more than one point and finishes the proof of the theorem.

Chapter 2

Fixed Points via Ultraproducts

2.0.4 Some preliminary result

In this chapter, we would like to give a proof of a fundamental result due to B. Maurey [12]. Maurey used ultraproduct techniques and the notion of random measures in his argument. One of his key ideas was that *half way between* every two a.f.p.s.'s there is another a.f.p.s..

Let us recall some notion that we have seen in the last chapter (see Lemma 1.21).

Theorem 2.1. *(Karlovitz)*

Let K be a weakly compact convex subset of a Banach space which is minimal for the non-expansive map $T : K \longrightarrow K$. Let $(x_n)_n$ be an a.f.p.s. for T and suppose (for simplicity) diam$K = 1$. Then for all $x \in K$

$$\lim_n \|x - x_n\| = 1.$$

Here we are in position to enunciate the key idea of Maurey's result.

Theorem 2.2. *Let K be a weakly compact convex subset of a Banach space which is minimal for the non-expansive map $T : K \longrightarrow K$. Let $(x_n)_n$ and $(y_n)_n$ be two a.f.p.s.'s for T. Suppose $\lim_n \|x_n - y_n\|$ exists (we can always assume this by passing to a subsequence, if necessary). Then there exists an a.f.p.s., $(z_n)_n$ for T such that*

$$\lim_n \|x_n - z_n\| = \lim_n \|y_n - z_n\| = \frac{1}{2} \lim_n \|x_n - y_n\| \tag{2.1}$$

Roughly speaking, this says that *halfway* between two points which are almost fixed by T there is a third point almost fixed by T.

Proof. We may assume $0 < \lambda = \lim_n \|x_n - y_n\|$. Fix $n \in \omega$ and choose $\varepsilon = \varepsilon(n)$ and δ so that $0 < \delta < 2\varepsilon^2$,

$$\|x_n - y_n\| \leq \lambda + \delta, \quad \|Tx_n - x_n\| < \frac{\delta}{2}, \quad \|Tx_n - x_n\| < \frac{\delta}{2}.$$

Let

$$\begin{aligned} K_n = \{z \in K : \|x_n - z\| \leq \frac{\lambda}{2} + \varepsilon \\ \|y_n - z\| \leq \frac{\lambda}{2} + \varepsilon\}. \end{aligned}$$

Then $(x_n + y_n)/2 \in K_n$, and K_n is a closed convex subset of K.

We claim the strict contraction

$$T_\varepsilon z = (1-\varepsilon)Tz + \varepsilon(x_n + y_n)/2$$

leaves K_n invariant.

Indeed, if $z \in K_n$, then $\|Tx_n - Tz\| \leq \|x_n - z\| \leq \lambda/2 + \varepsilon$ and hence

$$\begin{aligned} \|x_n - T_\varepsilon z\| &\leq (1-\varepsilon)\|x_n - Tz\| + \varepsilon\|x_n - (x_n + y_n)/2\| \\ &\leq (1-\varepsilon)[\|x_n - Tx_n\| + \|Tx_n - Tz\|] + \varepsilon\|(x_n - y_n)/2\| \\ &\leq (1-\varepsilon)(\frac{\delta}{2} + \frac{\lambda}{2} + \varepsilon) + \varepsilon(\frac{\lambda}{2} + \frac{\delta}{2}) \\ &< \frac{\lambda}{2} + \varepsilon^2 + \varepsilon(1-\varepsilon) \\ &= \frac{\lambda}{2} + \varepsilon. \end{aligned}$$

A similar estimate shows

$$\|y_n - T_\varepsilon z\| \leq \frac{\lambda}{2} + \varepsilon.$$

Thus by Banach-Caccioppoli's theorem, T_ε has a (unique) fixed point $z_n \in K_n$. Since

$$z_n = T_\varepsilon z_n = (1-\varepsilon)Tz_n + \varepsilon(x_n + y_n)/2,$$

we have

$$\|Tz_n - z_n\| = \|\varepsilon[Tz_n - (x_n + y_n)/2]\| \leq \varepsilon(\frac{\lambda}{2} + \varepsilon).$$

Note that $\varepsilon = \varepsilon(n)$ could be chosen so that $\varepsilon(n) \to 0$ as $n \to \infty$. Hence the resulting $(z_n)_n$ is an a.f.p.s. and (2.1) holds from the definition of K_n. □

2.0.5 A short introduction of Ultraproducts

Let I be a given index set

Definition 2.3. A *Filter* on I is a non-empty family of subsets $F \subseteq 2^I$ satisfying:

Fi) if $A, B \in F$, then $A \cap B \in F$

Fii) if $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$.

Example 2.4. (i) The improper filter $F = 2^I$

(ii) The indiscrete (trivial) filter $F = \{I\}$

(iii) For each $i_0 \in I$ the discrete filter at i_0, $F = \{A \subseteq I : \; i_0 \in A\}$.

A filter F is proper if $F \neq 2^I$. Note that a filter F is proper if and only if $\emptyset \notin F$ if and only if F has the finite intersection property.

If $S \subseteq 2^I$ is a non-empty family of subsets of I,

$$F_S = \{A \subseteq I : \text{ for some } S_1, \ldots, S_n \in S, \; S_1 \cap \ldots \cap S_n \subseteq A\}$$

is a filter on I containing S: it is called *filter generated by* S. If $B \subseteq 2^I$ is a non-empty family of subsets of I which is closed under finite intersections, then the filter generated by B can be written

$$F_B = \{A \subseteq I : \text{ for some } \overline{B} \in B, \; \overline{B} \subseteq A\}.$$

Definition 2.5. An *ultrafilter* is a filter which is maximal respect to the ordering by containment.

That is, a filter U is an ultrafilter if and only if whenever F is a proper filter with $U \subseteq F$, then $F = U$.

Lemma 2.6. *A filter U on I is an ultrafilter if and only if for every $A \subseteq I$ precisely one of the sets A and $I \setminus A$ belongs to U.*

Proof. We left the proof to the reader. □

The discrete filter at $i_0 \in I$ is an ultrafilter. We shall say that an ultrafilter is *trivial* if it is generated by a single element $i_0 \in I$.

Let X be a topological space and $(x_i)_{i \in I}$ be a family of elements of X indexed by I. Let U be an ultrafilter on I.

Definition 2.7. We say that $(x_i)_{i\in I}$ converges over U to x and we write

$$\lim_U x_i = x$$

if for every neighbourhood N of x

$$\{i \in I :\ x_i \in N\} \in U.$$

Note that if U is an non-trivial ultrafilter on the natural number ω, if $(x_n)_n$ converges, in the topology of X, to x then $\lim_U x_n = x$.

The assumption of non-trivial is essential. Indeed, if $U = F_{\{i_0\}}$ then for every family $(x_i)_i$,

$$\lim_U x_i = x_{i_0},$$

as for any neighbourhood N_0 of x_{i_0} we have $\{i_0\} \subseteq \{i \in I :\ x_i \in N_0\} \in U$.

Let $(A_i)_{i\in I}$ be given sets and let $\prod_i A_i$ denote their Cartesian product; that is the set of all functions

$$a : I \longrightarrow \bigcup_I A_i, \quad i \longmapsto a_i \in A_i.$$

It will be convenient to identify a with the family $(a_i)_i$.

Two families $(a_i)_i$ and $(b_i)_i$ are equivalent with respect to the ultrafilter U on I if

$$\{i \in I :\ a_i = b_i\} \in U,$$

and in such case we write $(a_i)_i \equiv_U (b_i)_i$. It is easy to show that $\equiv_U$ is an equivalent relation on $\prod_i A_i$. The equivalent class of $(a_i)_i$ will be denote by $(a_i)_U$

Definition 2.8. The set of all equivalent classes of $\prod_i A_i$ with respect to U is the (set-theoretic) *ultraproduct* of the family $(A_i)_{i\in I}$, which will be denoted by $\prod_i A_i/U$.

In the special case when all the A_i's are equal, to A say, their ultraproduct with respect to U may be written as $(A)_U$ and it is called ultraproduct of A with respect to U.

Let $(X_i)_{i\in I}$ be a family of Banach spaces and consider the Banach space $\ell_\infty(I, X_i)$ which consists of all families $(x_i)_i \in \prod_{i\in I} X_i$ such that

$$\|(x_i)_i\| = \sup_{i\in I} \|x_i\|_{X_i} < \infty.$$

If U is an ultrafilter on I, since $(\|x_i\|)_i$ is a bounded family of real numbers for each $(x_i)_i \in \ell_\infty(I, X_i)$, we see that $\lim_U \|x_i\|$ exists.

Let

$$N_U = \{(x_i)_i \in \ell_\infty(I, X_i) : \lim_U \|x_i\| = 0\}.$$

It is easy to see that N_U is a closed subspace of $\ell_\infty(I, X_i)$.

Definition 2.9. The *ultraproduct* of the family of Banach spaces $(X_i)_{i\in I}$ with respect to the ultrafilter U is the quotient space

$$(X_i)_U = \ell_\infty(I, X_i)/N_U$$

with the quotient norm

$$\|(x_i)_U\| = \inf\{\|(x_i) + (y_i)\| : (y_i) \in N_U\},$$

where $(x_i)_U$ is the equivalent class $(x_i) + N_U$.

Proposition 2.10. $\|(x_i)_U\| = \lim_U \|x_i\|$.

Proof. We first observe that for each $(y_i) \in N_U$,

$$\lim_U \|x_i + y_i\| = \lim_U \|x_i\| = r \text{ say.}$$

Indeed, given $\varepsilon > 0$ let $I_\varepsilon = \{i \in I : |\|x_i\| - r| < \varepsilon\}$, then $I_\varepsilon \in U$.

Since for $(y_i) \in N_U$,

$$I' = \{i \in I : |\|x_i + y_i\| - r| < \varepsilon\} \supseteq I_{\varepsilon/2} \cap \{i \in I : \|y_i\| < \varepsilon/2\} \in U$$

we see that $I' \in U$ as required.

From this it also follows that for $(y_i) \in N_U$

$$\sup_I \|x_i + y_i\| \geq \sup_{I'} \|x_i + y_i\| \geq r - \varepsilon$$

and so $\|(x_i)_U\| = \inf_{(y_i)\in N_U} \sup_I \|x_i + y_i\| \geq r.$.

To establish the opposite inequality, let I_ε be as above and define

$$y_i = \begin{cases} 0, & \text{for } i \in I_\varepsilon \\ -x_i, & \text{otherwise.} \end{cases}$$

Since $\{i \in I : \|y_i\| < \varepsilon_1\} \supseteq I_\varepsilon \in U$, then $(y_i) \in N_U$, and

$$\sup_I \|x_i + y_i\| = \sup_{I_\varepsilon} \|x_i\| < r + \varepsilon.$$

Thus, $\|(x_i)_U\| = \inf_{(y_i)\in N_U} \sup_I \|x_i + y_i\| \leq r.$ □

Let also notice that if U is the trivial ultrafilter $F_{\{i_0\}}$ then $(X_i)_U$ coincides with X_{i_0}.

If all the spaces X_i $(i \in I)$ are equal to a certain space X, the we refer to their ultraproduct respect to U as the *ultrapower* of X with respect to U, which we shall denote by $(X)_U$.

There is a canonical isometric embedding,

$$J : X \hookrightarrow (X)_U$$

given by

$$J(x) = (x_i)_U \quad \text{where } x_i = x \text{ for all } i \in I.$$

Therefore X is isometric to a closed subspace of $(X)_U$.

Let us recall some basic facts, where we leave the proofs to the reader.

Proposition 2.11. *Suppose that E_n is an n-dimensional Banach space for every $n \in \omega$ and that U is a (non-trivial) ultrafilter on ω. Then $(E_i)_U$ is non-separable.*

Proposition 2.12. *Let $(X_i)_{i\in I}$ be a family of Banach lattices. If U is an ultrafilter on I, then $(X_i)_U$ has a natural Banach lattice structure.*

In fact, given (x_i) and (y_i) in $\ell_\infty(I, X_i)$, define

$$(x_i)_U \leq (y_i)_U$$

whenever there is an element $(z_i) \in N_U$ such that $x_i + z_i \leq y_i$ for each $i \in I$

Theorem 2.13. *(a) Let $1 \leq p < \infty$. Ultraproducts of $L_p(\mu)$-spaces are isometrically isomorphic (as Banach lattices) to $L_p(\mu)$-spaces.*

(b) Ultraproducts of $C(K)$-spaces are isometrically isomorphic (as Banach lattices) to $C(K)$-spaces.

Proof. The proof rely on the following basic facts, due to a Kakutani:

For $1 \leq p < \infty$, a Banach lattice X is isometrically isomorphic (as a Banach lattice) to a $L_p(\mu)$-space if and only if $\|x + y\|^p = \|x\|^p + \|y\|^p$ for every $x, y \in X$ satisfying $x \wedge y = 0$.

A Banach lattice X is isometrically isomorphic (as a Banach lattice) to a $C(K)$-space if and only if $\|x \vee y\| = \|x\| \vee \|y\|$, for every $x, y \in X$, $x, y \geq 0$. □

Theorem 2.14. *Every Banach space X is isometrically isomorphic to a subspace of an ultraproduct of its finite dimensional subspaces.*

Definition 2.15. Let X and Y be Banach spaces, and let $\lambda \geq 1$ We say that Y is *λ-representable* in X if, no matter how we choose $\varepsilon > 0$, for each finite dimensional subspace F of Y we can find a finite dimensional subspace E of X and an isomorphism $u : F \longrightarrow E$ such that $\|u\| \cdot \|u^{-1}\| \leq \lambda + \varepsilon$. The case when $\lambda = 1$ is said *finite representable* in place of 1-representable.

Even if an ultrapower of an infinite dimension Banach space is always non-separable, we have a local representability

Theorem 2.16. *Let X be a Banach space. For every index set I and any ultrafilter U on I, $(X)_U$ is finitely representable in X.*

2.0.6 Maurey gets seriously

Once we have the concept of ultraproduct, we can translate Theorem 2.1 and Theorem 2.2 as:

let (as usual) K be a convex weakly compact subset of X which is minimal for the non-expansive map T. Suppose $diam K = 1$. Let

$$\widetilde{K} = \{(x_n) : \ x_n \in K \text{ for all } n\} \subseteq (X)_U,$$

and define

$$\widetilde{T} : \widetilde{K} \longrightarrow \widetilde{K}$$

by

$$\widetilde{T}(x_n) = (Tx_n).$$

Clearly $\widetilde{K}$ is closed and convex, and $\widetilde{T}$ is non-expansive on $\widetilde{K}$. Furthermore, $\widetilde{T}$ has fixed point in $\widetilde{K}$. Indeed, if $(x_n)_n$ is an a.f.p.s. for T in K, then

$$\|\widetilde{T}(x_n) - (x_n)\|_{(X)_U} = \lim_U \|Tx_n - x_n\| = \lim_n \|Tx_n - x_n\| = 0,$$

and hence $\widetilde{T}(x_n) = (x_n)$. Therefore we have

Theorem 2.17. *Let K be a convex weakly compact set of diameter 1 which is minimal for the non-expansive map T. Let $f = (x_n)$ be a fixed point of $\widetilde{T}$ in $\widetilde{K}$. Let $x \in K$ and $\widetilde{x} = (x, x, \ldots) \in \widetilde{K}$. Then*

$$\|\widetilde{x} - f\| = \lim_U \|x - x_n\| = 1.$$

Theorem 2.18. *Let K be a convex weakly compact set of diameter 1 which is minimal for the non-expansive map T. Let $f = (x_n)$ and $g = (y_n)$ be a fixed points for $\widetilde{T}$ in $\widetilde{K}$. then there is a fixed point $h = (z_n)$ such that*

$$\|f - h\|_{(X)_U} = \|g - h\|_{(X)_U} = \frac{1}{2}\|f - g\|_{(X)_U}.$$

Let us apply those Theorems when we treat reflexive subspace of $L_1[0,1]$.

Lemma 2.19. *Let X be a reflexive subspace of $L_1[0,1]$ and let $K \subseteq X$ be a convex weakly compact subset of diameter 1 which is minimal for the non-expansive map T. Regard $\widetilde{K} \subseteq L_1(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{P})$ (ultraproduct of L_1-spaces), and let $(f^\alpha)_{\alpha\in I}$ be a finite (or countable) collection of fixed points for $\widetilde{T}$. Then there exist two measurable functions U and V on $\widetilde{\Omega}$ such that for each $\alpha \in I$,*

$$\widetilde{P}(\{\omega \in \widetilde{\Omega} : \ f^\alpha(\omega) \neq U(\omega) \text{ and } f^\alpha(\omega) \neq V(\omega)\}) = 0.$$

For the proof of Lemma 2.19 we need two further sublemmas. For both sublemmas we assume $\widetilde{K}$ is as in the statement of Lemma 2.19. K is separable (since it is minimal) and thus it contains a dense sequence $(d_k)_k$. Recall that $\widetilde{d_k} = (d_k, d_k, d_k, \ldots) \in \widetilde{K}$.

Sublemma 2.20. *For each $f = (x_n) \in \widetilde{K}$ we have ($\widetilde{P}$-a.e.)*

$$\inf_k \widetilde{d_k} \leq f \leq \sup_k \widetilde{d_k}.$$

The inf *and* sup *here are taken pointwise on $\widetilde{\Omega}$.*

Proof. For $m \in \omega$, let $y_m = \sup_k d_k \wedge m$. This sup exists in the lattice L_1.

We claim that

$$\sup_k \widetilde{d_k \wedge m} = \widetilde{y_m}. \tag{2.2}$$

This sup is understood to be in the lattice $(L_1)_U$ (and thus (2.2) is valid pointwise $\widetilde{P}$-a.e. in $\widetilde{\Omega}$ as well). Indeed, $\leq$ is clear since for $k \in \omega$ $d_k \wedge m \leq y_m$ implies $\widetilde{d_k \wedge m} \leq \widetilde{y_m}$. The other inequality follows form the fact that

$$\sup_k d_k \wedge m \geq \bigvee_{k=1}^{j} d_k \wedge m \text{ and we call the last element } f^j,$$

where f^j increases $\widetilde{P}$-a.e. (and hence in norm) to $\widetilde{y_m}$.

Thus, pointwise $\widetilde{P}$-a.e. on $\widetilde{\Omega}$ we have

$$\begin{aligned}\sup_k \widetilde{d_k} &\geq \sup_{k,m} \widetilde{d_k \wedge m}\\ &= \sup_m [\sup_k \widetilde{d_k \wedge m}]\\ &= \sup_m \widetilde{y_m}\end{aligned}$$

$$\geq \sup_m (x_1 \wedge m, x_2 \wedge m, \ldots)$$
$$= (x_1, x_2, \ldots).$$

The last inequality holds since $y_m \geq x \wedge m$ for all $x \in K$, and the last equality follows form uniformly integrability of the sequence $(x_n)_n$ in L_1. This proves the right inequality in Sublemma. The proof of the left inequality is similar. □

Sublemma 2.21. *Let $f \in \widetilde{K}$ be a fixed point of $\widetilde{T}$ and let $x, y \in K$. Let $\widetilde{x} = (x, x, \ldots)$ and $\widetilde{y} = (y, y, \ldots)$. Then, $\widetilde{P}$-a.e., $\widetilde{x}(\omega)$ and $\widetilde{y}(\omega)$ lie both in $]-\infty, f(\omega)]$ or both in $[f(\omega), +\infty[$.*

Proof. By the triangle inequality, we have pointwise on $\widetilde{\Omega}$,

$$|f - (\widetilde{x} + \widetilde{y})/2| \leq \frac{1}{2}(|f - \widetilde{x}| + |f - \widetilde{y}|). \tag{2.3}$$

Hence by Theorem 2.17,

$$1 = \|f - (\widetilde{x} + \widetilde{y})/2\| \leq \frac{1}{2}(\|f - \widetilde{x}\| + \|f - \widetilde{y}\|) = 1.$$

In particular both side of (2.3) are of norm 1 and so are equal $\widetilde{P}$-a.e.. From this, the Sublemma follows directly. □

Proof. of Lemma 2.19.

It is suffices to show that any three fixed points of $\widetilde{T}$ take at most two distinct values at $\widetilde{P}$-almost all $\omega \in \widetilde{\Omega}$. Indeed, then set

$$U = \bigwedge_{\alpha \in I} f^\alpha \quad \text{and} \quad V = \bigvee_{\alpha \in I} f^\alpha.$$

Thus suppose, by contradiction, that f^1, f^2 and f^3 are fixed points of $\widetilde{T}$ and $\widetilde{A} \in \widetilde{\Sigma}$, $\widetilde{P}(\widetilde{A}) > 0$, with for all $\omega \in \widetilde{A}$,

$$f^1(\omega) < f^2(\omega) < f^3(\omega).$$

By Sublemma 2.20 applied to f^1 and f^3, we can find $\widetilde{B} \subseteq \widetilde{A}$, $\widetilde{P}(\widetilde{B}) > 0$ and $\widetilde{x} = (x, x, \ldots)$, $\widetilde{y} = (y, y, \ldots) \in \widetilde{K}$ so that

$$\widetilde{x}(\omega) < f^2(\omega) < \widetilde{y}(\omega),$$

for $\omega \in \widetilde{B}$. This contradicts Sublemma 2.21. □

Now we are ready to enunciate the main result of this section (see [12]).

Theorem 2.22. *(B. Maurey)*

Let X be a reflexive subspace of $L_1[0,1]$. Then X has the f.p.p.

Proof. Let f^{2^m} and f^0 be fixed point for $\widetilde{T}$ satisfying $\|f^{2^m} - f^0\| = 1$ (it is enough to consider a.f.p.s. (x_n) and take $f^{2^m} = (x_{2n})$ and $f^0 = (x_{2n-1})$). By iteration of Theorem 2.18, one can construct fixed points of $\widetilde{T}$ f^k for $1 \le k < 2^m$ so that

$$\begin{cases} \sum_{k=1}^{2^m} \|f^k - f^{k-1}\| = \|f^{2^m} - f^0\| = 1 \\ \|f^k - f^{k-1}\| = \frac{1}{2^m}, \text{ for } k = 1, 2, \ldots, m. \end{cases} \tag{2.4}$$

On the other hand we have, pointwise on $\widetilde{\Omega}$,

$$|f^{2^m}(\omega) - f^0(\omega)| \le \sum_{k=1}^{2^m} |f^k(\omega) - f^{k-1}(\omega)| \tag{2.5}$$

and hence by the first part of (2.4),

$$1 = \|f^{2^m} - f^0\| \le \|\sum_{k=1}^{2^m} |f^k - f^{k-1}|\| \le \sum_{k=1}^{2^m} \|f^k - f^{k-1}\| = 1.$$

In particular the L_1-norms of both sides of the inequality (2.5) are equal and so

$$|f^{2^m} - f^0| = \sum_{k=1}^{2^m} |f^k - f^{k-1}| \quad \widetilde{P} - a.e. \tag{2.6}$$

Apply Lemma 2.19 to the fixed points f^k, $k = 0, \ldots, 2^m$ to obtain U and V.

It follows for (2.6) that there exist disjoint measurable sets $\widetilde{A_k}$ for $1 \le k \le 2^m$ so that

$$|f^k - f^{k-1}| = |U - V| \chi_{\widetilde{A_k}}.$$

Thus $\{2^m(f^k - f^{k-1}),\ k = 1, \ldots, 2^m\}$ are normalized disjointly supported functions in $L_1(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{P})$ and hence isometrically span $\ell_1^{2^m}$. Thus ℓ_1 is finitely representable in $(X)_U$ and hence in X. Thus, since $X \subseteq L_1$, ℓ_1 embeds into X and so X is not reflexive (see [11]). □

Now, we shall investigate the fixed point property for non reflexive subspaces of $L_1[0,1]$.

Definition 2.23. We say that a Banach space $(X, \|\cdot\|_X)$ is *asymptotically isometric to* ℓ_1 if it has a normalized Schauder basis $(x_n)_n$ such that for some sequence $(\lambda_n)_n \subseteq]0, +\infty[$ increasing to 1, we have that

$$\sum_{n=1}^{\infty} \lambda_n |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\|_X \tag{2.7}$$

Theorem 2.24. *Let $(Y, \|\cdot\|_Y)$ be a Banach space containing an asymptotically isometric copy of ℓ_1. Then $(Y, \|\cdot\|_Y)$ fails the fixed point property.*

Proof. Let $(x_n)_n$ in Y and $(\lambda_n)_n$ satisfy (2.7) above. Now, fix a sequence $(\mu_n)_n$ satisfying

$\mu_n > \mu_{n+1}$

$\lim_n \mu_n = r > 0$.

Each $\mu_{n+1}/\mu_n \in]0, 1[$, so that by passing to a corresponding subsequence of $(x_n)_n$ and $(\lambda_n)_n$ (if necessary), we may assume that

$$\lambda_n > \frac{\mu_{n+1}}{\mu_n}, \quad \forall n \in \omega$$

Now, define $e_n = \mu_n x_n$, for all $n \in \omega$, and let

$$K = \{\sum_{n\in\omega} \alpha_n e_n : \ \alpha_n \geq 0, \ \sum_{n\in\omega} \alpha_n = 1\}.$$

Clearly K is closed and convex in Y. K is bounded since $\lim_n \mu_n = r > 0$.

Define

$$T : K \longrightarrow K$$

by

$$T(\sum_{n\in\omega} \alpha_n e_n) = \sum_{n\in\omega} \alpha_n e_{n+1}.$$

Of course, T is fixed point free on K. Finally, we show that T is non expansive on K.

Fix $z = \sum_{n\in\omega} \alpha_n e_n$ and $w = \sum_{n\in\omega} \beta_n e_n$ in K, with $z \neq w$. Then,

$$\begin{aligned} \|Tz - Tw\|_Y &= \left\| \sum_{n\in\omega} (\alpha_n - \beta_n) e_{n+1} \right\|_Y \leq \sum_{n\in\omega} |\alpha_n - \beta_n| \|e_{n+1}\| \\ &= \sum_{n\in\omega} |\alpha_n - \beta_n| \mu_{n+1} < \sum_{n\in\omega} |\alpha_n - \beta_n| \lambda_n \mu_n \end{aligned}$$

$$
\begin{aligned}
\text{(by (2.7))} \quad &\le \left\| \sum_{n\in\omega} (\alpha_n - \beta_n)\mu_n x_n \right\|_Y \\
&= \|z - w\|_Y .
\end{aligned}
$$

□

Immediately we have the following

Corollary 2.25. *Let $(X, \|\cdot\|_X)$ be a Banach space and Y be a subspace of X such that there exists a sequence $(v_n)_n \subseteq Y$, a sequence $(u_n)_n \subseteq X$ and a null sequence $(\gamma_n)_n \subseteq]0, +\infty[$ with the following properties*

(i) $\|\sum_{n=1}^{N} t_n u_n\|_X = \sum_{n=1}^{N} |t_n|$, *for all scalar sequences* $t_1, \ldots, t_N$ *and* $n \in \omega$*;*

(ii) $\|u_n - v_n\|_X < \gamma_n$, *for all* $n \in \omega$.

Then $(Y, \|\cdot\|_X)$ fails the fixed point property.

Proof. Without loss of generality, we can assume that each $\gamma_n < 1$ and $(v_n)_n$ is normalized. Then $(v_n)_n$ spans an asymptotically isometric copy of ℓ_1 in $(Y, \|\cdot\|_X)$ with the λ_n's in inequality (2.7) above given by $\lambda_n = 1 - \gamma_n$, for all $n \in \omega$. □

Theorem 2.26. *Every nonreflexive subspace of $L_1[0,1]$, with its usual norm, fails the fixed point property.*

Proof. We would like to use Corollary 2.25 for $X = L_1[0,1]$, Y a non reflexive subspace, showing that in this context we can always construct $(v_n)_n \subseteq Y$, $(u_n)_n \subseteq X$ and a null sequence $(\gamma_n)_n \subseteq]0, +\infty[$ satisfying the hypothesis of Corollary 2.25. □

Before to go on, we need the following

Lemma 2.27. *Let $(f_n)_n$ be a sequence of $L_1[0,1]$, and suppose that for each $\varepsilon > 0$ there exists a n_ε such that the set $\{t : |f_{n_\varepsilon}(t)| \ge \varepsilon \|f_{n_\varepsilon}\|_1\}$ has measure less than ε. Then $(f_n)_n$ has a subsequence $(g_n)_n$ such that $(g_n/\|g_n\|)_n$ is a basic sequence equivalent to ℓ_1's unit vector basis.*

Proof. Call $E = \{t : |f(t)| \ge \varepsilon \|f\|_1\}$. Suppose $\lambda(E) < \varepsilon$. Then

$$
\begin{aligned}
\int_E \frac{|f(t)|}{\|f\|} dt &= \int_0^1 \frac{|f(t)|}{\|f\|} dt - \int_{E^c} \frac{|f(t)|}{\|f\|} dt \\
&= 1 - \int_{\{|f(t)| < \varepsilon \|f\|\}} \frac{|f(t)|}{\|f\|} dt > 1 - \varepsilon.
\end{aligned}
$$

Therefore, under the hypotheses of the lemma, we can find E_1 and n_1 so that

$$\lambda(E_1) < \frac{1}{4^2}$$

and

$$\int_{E_1} \frac{|f_{n_1}(t)|}{\|f_{n_1}\|} dt > 1 - \frac{1}{4^2}.$$

Next, applying the hypotheses again and keeping the absolute continuity of the integral in mind, we can find E_2 and $n_2 > n_1$ so that

$$\lambda(E_2) < \frac{1}{4^3}$$

and

$$\int_{E_2} \frac{|f_{n_2}(t)|}{\|f_{n_2}\|} dt > 1 - \frac{1}{4^3}.$$

Continually applying such tactics, we generate a subsequence $(g_n)_n$ of $(f_n)_n$ and sets E_n such that

$$\int_{E_n} \frac{|g_n(t)|}{\|g_n\|} dt > 1 - \frac{1}{4^{n+1}}.$$

and

$$\int_{E_n} \sum_{k=1}^{n-1} \frac{|g_k(t)|}{\|g_k\|} dt < \frac{1}{4^{n+1}}.$$

Now we disjointify: let

$$A_n = E_n \setminus \cup_{k=n+1}^{\infty} E_k$$

and set

$$h_n(t) = \frac{g_n(t)}{\|g_n\|} \chi_{A_n}.$$

Therefore

$$\begin{aligned}
\left\| \frac{g_n}{\|g_n\|} - h_n \right\| &\leq \int_{A_n^c} \frac{g_n(t)}{\|g_n\|} dt \\
&\leq \int_{E_n^c} \frac{g_n(t)}{\|g_n\|} dt + \int_{E_n \setminus A_n} \frac{g_n(t)}{\|g_n\|} dt \\
&\leq \frac{1}{4^{n+1}} + \sum_{k=n+1}^{\infty} \int_{E_k} \frac{g_n(t)}{\|g_n\|} dt \\
&< \frac{1}{4^{n+1}} + \sum_{k=n+1}^{\infty} \frac{1}{4^{k+1}} < \frac{1}{4^n}
\end{aligned}$$

Thus

$$\begin{aligned} 1 \geq \|h_n\| &= \int_{A_n} \frac{g_n(t)}{\|g_n\|} dt \\ &\geq \int_{E_n} \frac{g_n(t)}{\|g_n\|} dt - \sum_{k=n+1}^{\infty} \int_{E_k} \frac{g_n(t)}{\|g_n\|} dt \\ &\geq 1 - \frac{1}{4^{n+1}} - \sum_{k=n+1}^{\infty} \frac{1}{4^{k+1}} \\ &> 1 - \frac{1}{4^n}. \end{aligned}$$

So

$$\begin{aligned} \left\| \frac{g_n}{\|g_n\|} - \frac{h_n}{\|h_n\|} \right\| &\leq \left\| \frac{g_n}{\|g_n\|} - h_n \right\| + \left\| h_n - \frac{h_n}{\|h_n\|} \right\| \\ &\leq \frac{1}{4^n} + (1 - \|h_n\|) \leq \frac{2}{4^n}. \end{aligned}$$

Notice that h_n's are disjointly supported non zero members of $L_1[0,1]$; therefore, $(h_n/\|h_n\|)_n$ is a basic sequence equivalent to the unit vector basis of ℓ_1. By [18, Porposition 5.4] we get that $(g_n/\|g_n\|)_n$ is a basic sequence equivalent to the unit vector basis of ℓ_1 too. □

Now, to finish the proof of the Theorem 2.26, we start with the nonweakly compact closed unit ball B_X of X. Let $0 < \mu \leq 1$ and set, for any $f \in L_1[0,1]$,

$$\alpha(f, \mu) = \sup \left\{ \int_E |f(t)| dt : \ \lambda(E) = \mu \right\}.$$

If $\alpha_X(\mu) = \sup_{f \in B_X} \alpha(f, \mu)$, then the non reflexivity of X is reflected by the conclusion that

$$\alpha^* = \lim_{\mu \to 0} \alpha_X(\mu) > 0.$$

Therefore, we can choose $f_n \in B_X$, measurable sets $E_n \subseteq [0,1]$, and $\mu_n > 0$ such that

$$\lim_n \mu_n = 0,$$

$$\int_{E_n} |f_n(t)| dt = \mu_n,$$

and

$$\lim_n \alpha(f_n, \mu_n) = \alpha^*.$$

Consider now the function f_n' given by

$$f_n'(t) = f_n(t)\chi_{E_n}.$$

Notice that given $\varepsilon > 0$ there is a n_ε so that

$$\lambda(\{t : \ |f_{n_\varepsilon}'(t)| \geq \varepsilon \|f_{n_\varepsilon}\|\}) < \varepsilon;$$

in other words, we have established the hypotheses of the previous lemma.

Combining with Theorem 2.22 we have that

Theorem 2.28. *Let Y be a subspace of $L_1[0,1]$ with its usual norm. Then the following are equivalent*

(i) Y is reflexive

(ii) Y has the fixed point property.

2.0.7 Fixed Points for Isometries

Let us recall that a Banach space X is called *superreflexive* if whenever Y is finitely representable in X, Y is reflexive. P. Enflo [8] proved that X is superreflexive if and only if X is isomorphic to a uniformly convex space. G. Pisier [14] strengthened Enflo's theorem by showing that if X is superreflexive, then there is an equivalent norm $|\cdot|$ on X, a number q, $2 \leq q < \infty$, and a $\gamma > 0$ such that for $x, y \in X$,

$$|\frac{x+y}{2}|^q \leq \frac{1}{2}(|x|^q + |y|^q) - \gamma^q |x-y|^q. \tag{2.8}$$

It is unknown whenever every superreflexive space has f.p.p.. B. Maurey solved the problem, however, for isometries.

Theorem 2.29. *Let K be a convex weakly compact subset of a superreflexive space X and let $T : K \longrightarrow K$ be an isometry. Thus $\|Tx - Ty\| = \|x - y\|$ for $x, y \in K$. Then T has a fixed point.*

Proof. Let $|\cdot|$ be an equivalent norm on X satisfying (2.8). For simplicity we assume that $q = 2$. Later we will indicate the necessary modification in the general case.. Thus we have

$$|\frac{x+y}{2}|^2 \leq \frac{1}{2}(|x|^2 + |y|^2) - \gamma^2 |x-y|^2, \quad \text{for } x, y \in X. \tag{2.9}$$

Also, as usual, we assume that K is minimal for T and $diam K = 1$.

We shall construct a function $\varphi : K \longrightarrow [0, M]$, for some $M < \infty$, satisfying

$$\varphi(\frac{1}{2}(x+y)) \geq \frac{1}{2}(\varphi(x)+\varphi(y)) + \|\frac{1}{2}(x-y)\|^2 \tag{2.10}$$

$$\varphi(Tx) \geq \varphi(x). \tag{2.11}$$

The equivalent norm $|\cdot|$ will be used in proving $\varphi(x) \leq M$, for some M.

Suppose such a φ has been constructed and let us complete the proof.

If φ achieved a maximum at some point $x_0 \in K$, then by (2.10) x_0 would be the unique maximum point and so by (2.11), $Tx_0 = x_0$. Since there is no a priori reason for φ to have a maximum, we must work a bit harder. Let $0 < \varepsilon < \frac{1}{4}$ and define

$$K_\varepsilon = \{x \in K,\ \varphi(x) \geq M_0 - \varepsilon\},$$

where $M_0 = \sup\{\varphi(x),\ x \in K\}$. If $x, y \in K_\varepsilon$ then by (2.10) $(x+y)/2 \in K_\varepsilon$. Thus K_ε is *dyadically convex.* This means if $x, y \in K_\varepsilon$ and α is a dyadic rational in $[0,1]$, $\alpha x + (1-\alpha)y \in K_\varepsilon$. It follows that $\overline{K_\varepsilon}$ is convex. By (2.11), $TK_\varepsilon \subseteq K_\varepsilon$ and hence $\overline{K_\varepsilon}$ is also invariant under T. Furthermore, by (2.10), if $x, y \in K_\varepsilon$ then $\|x-y\| \leq 2\varepsilon^{\frac{1}{2}}$. Thus $diam\overline{K_\varepsilon} < 1$. but then $\overline{K_\varepsilon}$ is a proper convex weakly compact subset of K which is invariant under T, so K is not minimal for T which is a contradiction.

To define $\varphi : K \longrightarrow [0, M]$ will require some notation. Let $\widetilde{X}$ be an ultrapower of X. Note that (2.9) holds also for $x, y \in \widetilde{X}$. Fix $f \in \widetilde{K}$ so that $\widetilde{T}f = f$, let $y \in K$ and let $\mathcal{D}$ be the dyadic rationals on $[0,1]$. A <u>configuration</u>, C, about y is a collection of points in $\widetilde{K}$,

$$C = (y^r_{\underline{i}})^{r \in \mathcal{D}}_{\underline{i} \in \{0,1\}^\omega}$$

satisfying

$$y^0_{\underline{i}} = y, \quad y^1_{\underline{i}} = f$$

and such that for $n \in \omega$ and $0 \leq k \leq 2^n$,

(i) $y^{k2^{-n}}_{\underline{i}} = y^{k2^{-n}}_{\underline{j}}$ if $\underline{i}|_n = \underline{j}|_n$;

(ii) $y^{(2k+1)2^{-n}}_{\underline{i}}$ is a metric midpoint of $y^{k2^{-n+1}}_{\underline{i}}$ and $y^{(k+1)2^{-n+1}}_{\underline{i}}$.

Perhaps this requires some explanation. If $\underline{i} = (i_1, i_2, \ldots) \in \{0,1\}^\omega$, then $\underline{i}|_n = (i_1, i_2, \ldots, i_n)$. *(ii)* says that

$$\|y^{(2k+1)2^{-n}}_{\underline{i}} - y^{k2^{-n+1}}_{\underline{i}}\| = \|y^{(2k+1)2^{-n}}_{\underline{i}} - y^{(k+1)2^{-n+1}}_{\underline{i}}\|.$$

Note that by (i) it makes sense to speak of $y_{\underline{i}}^{k2^{-n}}$ for $\underline{i} \in \{0,1\}^n$ - i.e., the tail of $\underline{i}$ past the n-th place has no effect on the element. The point of this apparent complication in notation is to simplify (ii).

By Theorem 2.17, since $\|f-y\|=1$, we can also use (ii) to calculate the distance (in $\|\cdot\|$-norm) between points connected by lines.

Associated with a configuration $C=(y_{\underline{i}}^r)$ about y is a family of non-negative reals,

$$\Delta(C)=(\delta_{i_1,\dots,i_{n-1}}^{(2k-1)2^{-n}})_{n\in\omega,\ 1\le k\le 2^{n-1}},$$

where $i_j=0$ or 1 for each j. These numbers are defined by

$$\delta_{i_1,\dots,i_{n-1}}^{(2k-1)2^{-n}}=\|y_{i_1,\dots,i_{n-1},0}^{(2k-1)2^{-n}}-y_{i_1,\dots,i_{n-1},1}^{(2k-1)2^{-n}}\|.$$

Thus, for example

$$\delta^{\frac{1}{2}}=\|y_0^{\frac{1}{2}}-y_1^{\frac{1}{2}}\|,\quad \text{and } \delta_0^{\frac{3}{4}}=\|y_{0,0}^{\frac{3}{4}}-y_{0,1}^{\frac{3}{4}}\|.$$

We define the <u>width</u> of the configuration C by

$$W(C)=\sum_{\delta\in\Delta(C)}\delta^2.$$

For $y\in K$, define

$$\varphi(y)=\sup\{W(C):\ C \text{ is a configuration about } y\}.$$

We must first check that there is an $M<\infty$ so that $\varphi(y)\le M$ for $y\in K$. This is where we need the equivalent norm $|\cdot|$ which satisfy (2.9).

Lemma 2.30. *Let $A,B,C,D\in\widetilde{X}$. Then*

$$4\gamma^2|D-B|^2\le|A-B|^2+|B-C|^2+|C-D|^2+|D-A|^2-|A-C|^2. \tag{2.12}$$

Proof. Let us rewrite (2.9) as

$$\gamma^2|x-y|^2\le\frac{1}{2}(|x|^2+|y|^2)-|(x+y)/2|^2. \tag{2.13}$$

Let $M=(C+A)/2$. We first wish to estimate $|M-D|$. Since $2(M-D)=(C-D)-(D-A)$ and $(C-A)/2=C-M$, (2.13) yelds

$$4\gamma^2|M-D|^2\le\frac{1}{2}(|C-D|^2+|D-A|^2)-|C-M|^2,$$

or

$$8\gamma^2|M-D|^2\le|C-D|^2+|D-A|^2-2|C-M|^2. \tag{2.14}$$

Similarly

$$8\gamma^2|M-B|^2 \le |A-B|^2 + |B-C|^2 - 2|A-M|^2. \tag{2.15}$$

Since $|A-C|^2 = 2|C-M|^2 + 2|A-M|^2$, combining (2.14) and (2.15) yelds

$$8\gamma^2(|M-D|^2+2|M-B|^2) \le |A-B|^2+|B-C|^2+|C-D|^2+|D-A|^2-|A-C|^2.$$

But $|B-D|^2 \le 2|M-D|^2 + 2|M-B|^2$ and so (2.12) follows. □

Let $C = (y^r_{\underline{i}})$ be any configuration about $y \in K$ and let $\Delta(C) = (\delta^{(2k-1)2^{-n}}_{i_1,\ldots,i_{n-1}})$. Define

$$\Delta'(C) = \Delta(C) = (\beta^{(2k-1)2^{-n}}_{i_1,\ldots,i_{n-1}})$$

by

$$\beta^{(2k-1)2^{-n}}_{i_1,\ldots,i_{n-1}} = |y^{(2k-1)2^{-n}}_{i_1,\ldots,i_{n-1},0} - y^{(2k-1)2^{-n}}_{i_1,\ldots,i_{n-1},1}|.$$

Since the norm $|\cdot|$ and $\|\cdot\|$ are equivalent, there is a constant $\lambda < \infty$ so that

$$\lambda^{-1}\|x\| \le |x| \le \lambda\|x\|, \quad \text{for } x \in \widetilde{X}. \tag{2.16}$$

We must show $W(C) \le M$ for some $M < \infty$ independent of y and C. It suffices by (2.16), to show that $4\gamma^2 \sum_{\beta\in\Delta'(C)} \beta^2$ is bounded by the number $\lambda^2 - |f-y|^2$ which is in turn $\le \lambda^2$.

Fix $n \in \omega$ and consider

$$4\gamma^2 \sum_{\beta\in\Delta'_m(C)} \beta^2 \text{ where } \Delta'_m(C) = \{\beta^{(2k-1)2^{-n}}_{i_1,\ldots,i_{n-1}} : \ n \le m\}.$$

Iteration of Lemma 2.30 yields the desired result. Consider, for example, $m = 2$. By Lemma 2.30,

$$\begin{aligned}(\beta^{\frac{1}{2}})^2 \le\ & |y^1 - y_1^{\frac{1}{2}}|^2 + |y_1^{\frac{1}{2}} - y^0|^2 \\ & + |y_0^{\frac{1}{2}} - y^0|^2 + |y_0^{\frac{1}{2}} - y^1|^2 - |y^1 - y^0|^2.\end{aligned}$$

and

$$\begin{aligned}(\beta^{\frac{3}{4}})^2 \le\ & |y^1 - y_{11}^{\frac{3}{4}}|^2 + |y_{11}^{\frac{3}{4}} - y_1^{\frac{1}{2}}|^2 \\ & + |y_{10}^{\frac{3}{4}} - y_1^{\frac{1}{2}}|^2 + |y^1 - y_{11}^{\frac{3}{4}}|^2 - |y^1 - y_1^{\frac{1}{2}}|^2,\end{aligned}$$

and so forth. Thus by the telescoping property of the ensuring series we obtain,

$$4\gamma^2[(\beta^{\frac{1}{2}})^2 + (\beta_1^{\frac{3}{4}})^2 + (\beta_1^{\frac{1}{4}})^2 + (\beta_0^{\frac{1}{4}})^2 + (\beta_0^{\frac{3}{4}})^2]$$

$$\leq 4\gamma^2[-|y^1 - y^0|^2 + (|y^1 - y_{11}^{\frac{3}{4}}|^2 + \ldots + |y_{00}^{\frac{3}{4}} - y^1|^2)].$$

There are sixteen terms in the parentheses, and if we estimate $|\cdot|^2$ by $\lambda^2\|\cdot\|^2$ (see (2.16)) for each term we get

$$\begin{aligned}&\leq 4\gamma^2[-|f-y|^2 + \lambda^2((\frac{1}{4})^2 + (\frac{1}{4})^2 + \ldots + (\frac{1}{4})^2)]\\&= 4\gamma^2(\lambda^2 - |f-y|^2).\end{aligned}$$

An obvious modification of this argument works for any m. It remains only to verify (2.10) and (2.11).

$\varphi(Tx) \geq \varphi(x)$ follows from the fact that if $(x_{\underline{i}}^r)$ is a configuration about x, then $(\widetilde{T}x_{\underline{i}}^r)$ is a configuration about Tx of the same width. Here we are using that $\widetilde{T}$ is an isometry and thus preserves the width of a configuration. Note that $\widetilde{T}$ would preserve a configuration even if it were only a contraction.

It remains to show (2.10). This is slightly more complicated. We claim that if $(x_{\underline{i}}^r) = C_1$ is a configuration about x and $(y_{\underline{i}}^r) = C_2$ is a configuration about y, then there is a configuration $C = (z_{\underline{i}}^r)$ about $(x+y)/2$ with

$$W(C) = \frac{1}{2}(W(C_1) + W(C_2)) + \|\frac{x-y}{2}\|^2 \tag{2.17}$$

which certainly implies (2.10). Indeed, define for $r \in \mathcal{D}$

$$\begin{cases} z_{0,\underline{i}}^{(r+1)/2} = \frac{1}{2}(x^1 + x_{\underline{i}}^r)\\ z_{0,\underline{i}}^{r/2} = \frac{1}{2}(x^0 + y_{\underline{i}}^r)\\ z_{1,\underline{i}}^{(r+1)/2} = \frac{1}{2}(y^1 + y_{\underline{i}}^r)\\ z_{1,\underline{i}}^{r/2} = \frac{1}{2}(y^0 + x_{\underline{i}}^r).\end{cases}$$

It is easily checked that $(z_{\underline{i}}^r)$ is a configuration about $(x+y)/2$. the key property of this configuration is that when one computes its width, the δ^2 terms of $W(C_1)$ and $W(C_2)$ are now divided by four but each occurs twice.

Also $W(C)$ contains an extra term, $\|z_0^{\frac{1}{2}} - z_1^{\frac{1}{2}}\|^2 = \|\frac{x-y}{2}\|^2$. Thus (2.17) holds and the proof of the theorem is complete in case $q = 2$ (in (2.9)).

The general argument is essentially the same except we cannot just define $W(C) = \sum_{\delta\in\Delta(C)} \delta^q$. We could not prove (2.10) with this definition. Instead, we must us the weights and the define

$$W_q(C) = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \sum_{(i_1,\ldots,i_{n-1})\in\{0,1\}^{n-1}}^{2^{n(q-2)}} (\delta_{i_1,\ldots,i_{n-1}}^{(2k-1)2^{-n}})^q.$$

The argument is then the same (except for some obvious modifications) as the one given in the case $q = 2$. □

2.0.8 Fixed Points and Unconditional Basis

Let X be a Banach space. Recall that a sequence $\{e_n\}_n$ in X is called *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_n$ so that

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

A Schauder basis $\{e_n\}_n$ is called *unconditional basis* if for any choice of signs ε_n (i.e. $\varepsilon_n = \pm 1$), $\sum_{n=1}^{\infty} \varepsilon_n a_n e_n$ converges whenever $\sum_{n=1}^{\infty} a_n e_n$ converges. If $\{e_n\}_n$ is an unconditional basis, then the number

$$\sup \left\{ \left\| \sum_{n=1}^{\infty} \varepsilon_n a_n e_n \right\| : \left\| \sum_{n=1}^{\infty} a_n e_n \right\| = 1, \ \varepsilon_n = \pm 1 \right\}$$

is called the *unconditional constant* of $\{e_n\}_n$. If $\{e_n\}_n$ is an unconditional basis and F is a subset of ω, then the projection

$$P_F(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n \in F} a_n e_n$$

is called the *natural projection* associate with F to the unconditional basis $\{e_n\}_n$. It is clear that the norm of any natural projections is smaller than the unconditional constant of the basis. We say that an unconditional basis is *suppressed unconditional* if every natural projection associate to the basis has norm 1.

Example 2.31. Let X_M be ℓ_2 with the new norm

$$\|x\| = \max\{\|x\|_\infty, \ M^{-1}\|x\|_2\}.$$

Then the natural basis is unconditional basis with unconditional constant $\lambda = 1$. It is known that X_M fails to have normal structure (see the section on Karlovitz's construction) whenever $M \geq \sqrt{2}$. But X_M still have the fixed point property.

Let us recall a variant of Theorem 2.1 in terms of ultraproduct

Theorem 2.32. *Let K be a minimal weakly compact convex set for a non expansive map T. If $\widetilde{y}$ is a fixed point of $\widetilde{T}$ in $\widetilde{K}$ and $x \in K$, then $\|\widetilde{y} - x\| = diam(K)$. Moreover, suppose $diam(K) = 1$ and $0 \in K$, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\widetilde{y}\| > 1 - \varepsilon$ whenever $\|\widetilde{T}\widetilde{y} - \widetilde{y}\| < \delta$.*

Finally, recall that two sequences $(x_n)_n$ and $(y_n)_n$ are called *disjoint* if x_n and y_n are disjoint in X for all $n \in \omega$. That means there exist natural projections $\widetilde{P} = (P_n)_n$ and $\widetilde{Q} = (Q_n)_n$ with respect to $(e_n)_n$ such that

$$\widetilde{P}\widetilde{x} = \widetilde{x}, \ \widetilde{Q}\widetilde{y} = \widetilde{y}$$

and

$$\widetilde{P}\widetilde{Q} = \widetilde{Q}\widetilde{P} = 0.$$

Now, we are ready to enunciate the main theorem of this section.

Theorem 2.33. *Every Banach space X with 1-unconditional basis $\{e_n\}_n$ has the weakly fixed point property.*

Proof. Suppose it were not true. Then there is a weakly compact convex subset K which is minimal for a nonexpansive map T, with $diam(K) = 1$.

By translation of K, then passing to a subsequence, we may suppose that $0 \in K$ and there exists an approximate fixed point sequence $(x_n)_n$ for T and natural projections P_n on X (with respect to $(e_n)_n$) such that

$P_nP_m = 0$ if $n \neq m$;

$\lim_{n\to\infty} \|P_nx_n\| = \lim_{n\to\infty} \|x_n\| = 1$;

$\lim_{n\to\infty} \|(I - P_n)x_n\| = 0$.

Let $\widetilde{y} = (x_n)_n$ and $\widetilde{z} = (x_{n+1})_n$. Then $\widetilde{y}$ and $\widetilde{z}$ are fixed points of $\widetilde{T}$ with $\|\widetilde{y} - \widetilde{z}\| = 1$. For any $x \in K$, $x, \widetilde{y}$ and $\widetilde{z}$ are disjoints.

Indeed, let $\widetilde{P} = (P_n)_n$ and $\widetilde{Q} = (P_{n+1})_n$. Then $\widetilde{P}\widetilde{y} = \widetilde{y}$ and $\widetilde{Q}\widetilde{z} = \widetilde{z}$ and for any $x \in K$,

$$\widetilde{P}x = \widetilde{Q}x = \widetilde{P}\widetilde{z} = 0 = \widetilde{Q}\widetilde{y}.$$

Also, since $(e_n)_n$ is 1-unconditional, $\|\widetilde{y} - \widetilde{z}\| = 1 = \|\widetilde{y} + \widetilde{z}\|$. Let

$$\widetilde{W} = \{\widetilde{w} \in \widetilde{K} : \text{ such that there exists } x \in K \text{ (depending on } \widetilde{w}) \text{ with } \max\{\|\widetilde{w} - x\|, \ \|\widetilde{w} - \widetilde{y}\|, \ \|\widetilde{w} - \widetilde{z}\|\} \leq 1/2\}.$$

Clearly, $\widetilde{W}$ is a nonempty bounded closed convex set. Since $\widetilde{y}$ and $\widetilde{z}$ are fixed points of $\widetilde{T}$ and T is a nonexpansive mapping, if $\widetilde{w} \in \widetilde{W}$,

$$\max\{\|\widetilde{T}\widetilde{w} - Tx\|, \ \|\widetilde{T}\widetilde{w} - \widetilde{y}\|, \ \|\widetilde{T}\widetilde{w} - \widetilde{z}\| \leq \max\{\|\widetilde{w} - Tx\|, \ \|\widetilde{w} - \widetilde{y}\|, \ \|\widetilde{w} - \widetilde{z}\| \leq \frac{1}{2}.$$

Thus $\widetilde{W}$ is invariant under $\widetilde{T}$, hence, it contains an approximate fixed point sequence for $\widetilde{T}$. On the other hand, for any $\widetilde{w} \in \widetilde{W}$ there exists $x \in K$ so that $\|\widetilde{w} - x\| \leq 1/2$. Hence if $\widetilde{I}$ is the identity map in $\widetilde{X}$,

$$\begin{aligned}
\|\widetilde{w}\| &= \frac{1}{2}\|(\widetilde{P} + \widetilde{Q})\widetilde{w} + (\widetilde{I} - \widetilde{P})\widetilde{w} + (\widetilde{I} - \widetilde{Q})\widetilde{w}\| \\
&\leq \frac{1}{2}[\|(\widetilde{P} + \widetilde{Q})\widetilde{w}\| + \|(\widetilde{I} - \widetilde{P})\widetilde{w}\| + \|(\widetilde{I} - \widetilde{Q})\widetilde{w}\|] \\
&= \frac{1}{2}[\|(\widetilde{P} + \widetilde{Q})(\widetilde{w} - x)\| + \|(\widetilde{I} - \widetilde{P})(\widetilde{w} - \widetilde{y})\| + \|(\widetilde{I} - \widetilde{Q})(\widetilde{w} - \widetilde{z})\|] \\
&\frac{1}{2}[\frac{1}{2} + \frac{1}{2} + \frac{1}{2}] = \frac{3}{4}.
\end{aligned}$$

By Theorem 2.32, $\widetilde{W}$ cannot contain any approximate fixed point sequences for $\widetilde{T}$. We have a contradiction. □

For a suppression unconditional basis, we have

Theorem 2.34. *Suppose X has a suppression unconditional basis $(e_n)_n$. Then X has the fixed point property whenever X is superreflexive.*

Proof. Suppose not and, as usual, let K be a minimal set of diameter 1 for a nonexpansive map T. Let $\widetilde{x}_1, \ldots, \widetilde{x}_n$ be disjoint fixed points for $\widetilde{T}$ in $\widetilde{K}$. We shall prove $(\widetilde{x}_i)_{i=1}^n$ is 2-equivalent to the unit basis of ℓ_1^n. Indeed, if $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$ and $0 < c < 1$, then the same argument as given before shows that every element in

$$\begin{aligned}
\widetilde{W} = \{\widetilde{w} \in \widetilde{K} :\ &\text{such that there exists } x \in K \\
&\text{with } \|\widetilde{w} - x\| \leq c, \text{ and } \|\widetilde{w} - \widetilde{x}_i\| \leq 1 - \alpha_i, \text{ for } i = 1, \ldots, n\}
\end{aligned}$$

has norm less than or equal to $1 - (1-c)/n$. $\widetilde{W}$ is a closed convex set which is invariant under $\widetilde{T}$; hence, $\widetilde{W}$ is empty. But

$$\left\|\widetilde{x}_j - \sum_{i=1}^n \alpha_i \widetilde{x}_i\right\| = \left\|\sum_{i \neq j} \alpha_i (\widetilde{x}_j - \widetilde{x}_i)\right\| \leq 1 - \alpha_j,$$

for $j = 1, 2, \ldots, n$. Thus

$$\|\sum_{i=1}^n \alpha_i \widetilde{x}_i\| > c \text{ and so } \|\sum_{i=1}^n \alpha_i \widetilde{x}_i\| = 1.$$

□

Now, we would like to generalize the above result and make it clearer. First, let us recall some stuff about basis.

Definition 2.35. (1) A sequence $(x_n)_n$ in a Banach space X is called *basic sequence* if it is a basis for its closed linear span, $\overline{span}\{x_n,\ n \in \omega\}$; that is, if for each $x \in \overline{span}\{x_n,\ n \in \omega\}$ one can find a unique sequence of scalars $(a_n)_n$ such that the series $\sum_n a_n x_n$ converges to x.

(2) Let $(x_n)_n$ be a basic sequence in a Banach space X. A sequence of non-zero vectors $u_m)_m$ in X of the form

$$u_m = \sum_{n=p_m+1}^{p_{m+1}} a_n x_n$$

where (a_n) are scalars and $p_1 < p_2 < \cdots$ is an increasing sequence of integers, is called a *block basic sequence* or more briefly a *block basis* of $(x_n)_n$.

Throughout the following, we shall denote by P_n for $P_{[0,n]}$, the natural projection associate to the basis $(e_n)_n$ through the subset $\{0, 1, \dots, n\}$ of ω.

Proposition 2.36. *Let $(e_n)_n$ be a normalized Schauder basis of X with associate biorthogonal system $(e_n^*)_n$. Let $(x_k)_k$ be a bounded sequence such that*

$$e_n^*(x_k) \longrightarrow 0 \quad \text{as } k \to \infty.$$

Then there is a subsequence $(x_{k_i})_i$ of $(x_k)_k$ and a sequence $(u_i)_i$ of successive blocks of $(e_n)_n$ such that

$$\lim_i \|x_{k_i} - u_i\| = 0.$$

Proof. Let $(\epsilon_i)_i$ be a sequence of positive numbers going to 0. One can find $N_0 \in \omega$ such that $\|x_{N_0} - P_{N_0} x_{N_0}\| \le \epsilon_0$. Since

$$\lim_n \|P_{N_0} x_n\| = 0,$$

one can find $n_1 > N_0$ such that $\|P_{N_0} x_n\| \le \epsilon_1$ for any $n \ge n_1$. Then let $N_1 > N_0$ satisfying $\|x_{n_1} - P_{N_1} x_{n_1}\| \le \epsilon_1$. Since

$$\lim_n \|P_{N_1} x_n\| = 0,$$

one can find $n_2 > N_1$ such that $\|P_{N_1} x_n\| \le \epsilon_2$ for any $n \ge n_2$. Then let $N_2 > N_1$ satisfying $\|x_{n_2} - P_{N_2} x_{n_2}\| \le \epsilon_2$. Prossing in this way, we are constructing a sequence of pairs $\{(n_i, N_i)\}_i$ with $n_1 < n_2 \dots$ and $N_1 < N_2 < \dots$ such that

$$\|P_{N_{k-1}} x_n\| \le \epsilon_k \quad \text{for } n \ge n_k$$

and

$$\|x_{n_k} - P_{N_k} x_{n_k}\| \leq \epsilon_k.$$

Let us put $u_k = (I - P_{n_k} + P_{n_{k-1}})x_{n_k}$ for $k \in \omega$. We obtain

$$\|x_{n_k} - u_k\| \leq \epsilon_{k-1} + \epsilon_k.$$

The support of u_k is clearly in the interval $[N_{k-1}, N_k]$. This complete the proof. □

Let X be a Banach space with a Schauder basis $(e_n)_n$. Let $(x_n)_n$ be a sequence which converges weakly to zero in X. Using the above proposition, one can find a subsequence $(x'_n)_n$ of $(x_n)_n$ and a sequence of natural projections $(P_{F_n})_n$, where $(F_n)_n$ is a sequence of disjoint successive intervals of ω, such that

$$\lim_n \|P_{F_n}(x'_n) - x'_n\| = 0. \tag{2.18}$$

If we denote P_{F_n} by P_n, we can use the properties of $(F_n)_n$ to deduce the following

$$P_n \circ P_m = 0 \quad \text{if } n \neq m; \tag{2.19}$$

$$\lim_n \|P_n(x)\| = 0 \quad \text{for any } x \in X. \tag{2.20}$$

We associate new constants to the Schauder basis as follows:

$$\mu = \sup\{\|u - v\| : \ u \text{ and } v \text{ are disjoint block on } (e_n)_n \text{ with } \|u + v\| \leq 1\}$$

$$c_1 = \sup\{\|I - P_n\| : \ n \ in \omega\}$$

$$c_2 = \sup\{\|I - P_F\| : \ F \text{ is any segment of } \omega\}$$

$$c = \sup\{\|P_n\| : \ n \in \omega\}.$$

Here is a more general theorem which include one already seen above.

Theorem 2.37. *Let X be a Banach space with a Schauder basis $(e_n)_n$. Assume that the constants μ, c_1, c_2, c satisfy*

$$c_1\mu + c + c_2 < 4;$$

then X has the weak fixed point property.

Proof. Assume that X fails to have w.f.p.p., so there exists a nonempty weakly compact convex subset C of X and a non expansive mapping $T : C \longrightarrow C$ with $Fix(T) = \emptyset$.

Let K be a minimal set for T; without loss of generality, we can assume that $diam K = 1$.

Let $(x_n)_n$ be an a.f.p.s. in K for T. Since K is weakly compact, we can assume that $(x_n)_n$ is weakly convergent. Also, since the fixed point problem is invariant under translation, we can assume that $(x_n)_n$ is weakly null. Let $(P_n)_n$ as above satisfying (2.18), (2.19) and (2.20). Moreover, by Lemma 1.21, we can assume that

$$\lim_n \|x_{n+1} - x_n\| = 1 \tag{2.21}$$

Let $X^{\mathcal{U}}$ be an ultrapower of X, whenever $\mathcal{U}$ is a non trivial ultrafilter on ω, and let $\widetilde{K}$ and $\widetilde{T}$ defined as always. Consider

$$\widetilde{x} = (x_n)_n \text{ and } \widetilde{y} = (x_{n+1})_n \text{ in } \widetilde{K}.$$

Clearly $\widetilde{x}$ and $\widetilde{y}$ are fixed points for $\widetilde{T}$. Define the operators:

$$\widetilde{P} = (P_n)_{\mathcal{U}} \text{ and } \widetilde{Q}(I - \widehat{P}_n)_{\mathcal{U}}$$

where $\widehat{P}_n$ is the projection on $[1, maxF_n]$.

By construction, we obtain

$$\widetilde{P}(\widetilde{x}) = \widetilde{x},\ \widetilde{Q}(\widetilde{y}) = \widetilde{y}$$

and

$$\widetilde{P}(\widetilde{x}) = \widetilde{Q}(\widetilde{x}) = \widetilde{P}(x) = \widetilde{Q}(x) = 0,$$

for all $x \in X$. Moreover, by (2.21), we have

$$\|\widetilde{x} + \widetilde{y}\| = \|\widetilde{P}(\widetilde{x}) + \widetilde{Q}(\widetilde{y})\| = \lim_{\mathcal{U}} \|P_n(x_n) + Q_n(x_{n+1})\|.$$

But

$$\|P_n(x_n) + Q_n(x_{n+1})\| \leq \mu\|P_n(x_n) - Q_n(x_{n+1})\|,$$

therefore

$$\|\widetilde{x} + \widetilde{y}\| \leq \mu\|\widetilde{P}(\widetilde{x}) - \widetilde{Q}(\widetilde{y})\| = \mu\|\widetilde{x} - \widetilde{y}\| = \mu. \tag{2.22}$$

Using the definitions of $\widetilde{P}$ and $\widetilde{Q}$ we obtain

$$\|\widetilde{P} + \widetilde{Q}\| \leq c_1,\ \|I - \widetilde{P}\| \leq c_2, \text{ and } \|I - \widetilde{Q}\| \leq c.$$

Now set

$$\widetilde{W} = \{\widetilde{w} \in \widetilde{K} : \text{ such that there exists } x \in K \text{ such that}$$
$$\|\widetilde{w} - x\| \leq \frac{\mu}{2},\ \|\widetilde{w} - \widetilde{x}\| \leq \frac{1}{2},\ \|\widetilde{w} - \widetilde{y}\| \leq \frac{1}{2}\}.$$

$\widetilde{W}$ is a closed convex subset of $\widetilde{K}$. Using (2.22) we deduce that

$$\frac{\widetilde{x}+\widetilde{y}}{2} \in \widetilde{W}, \text{ since } 0 \in \widetilde{K}.$$

It is easy that $\widetilde{W}$ is invariant under $\widetilde{T}$, since $\widetilde{T}x = Tx$ whenever $x \in K$ and $\widetilde{x}, \widetilde{y}$ are fixed points for $\widetilde{T}$.

Let $\widetilde{w} \in \widetilde{W}$ and $x \in K$ such that $\|\widetilde{w} - x\| \leq \mu/2$. Then

$$\begin{aligned} 2\widetilde{w} &= (\widetilde{P}+\widetilde{Q})\widetilde{w} + (\widetilde{I}-\widetilde{P})\widetilde{w} + (\widetilde{I}-\widetilde{Q})\widetilde{w} \\ &= (\widetilde{P}+\widetilde{Q})(\widetilde{w}-x) + (\widetilde{I}-\widetilde{P})(\widetilde{w}-\widetilde{x}) + (\widetilde{I}-\widetilde{Q})(\widetilde{w}-\widetilde{y}), \end{aligned}$$

so that

$$\begin{aligned} 2\|\widetilde{w}\| &\leq \|\widetilde{P}+\widetilde{Q}\|\|\widetilde{w}-x\| + \|\widetilde{I}-\widetilde{P}\|\|\widetilde{w}-\widetilde{x}\| + \|\widetilde{I}-\widetilde{Q}\|\|\widetilde{w}-\widetilde{y}\| \\ &\leq c_1\frac{\mu}{2} + c_2\frac{1}{2} + c\frac{1}{2}. \end{aligned}$$

Hence

$$\sup\{\|\widetilde{w}\| : \widetilde{w} \in \widetilde{W}\} \leq \frac{\mu c_1 + c_2 + c}{4} < 1.$$

Now, by a classical argument, let us consider an a.f.p.s. $(\widetilde{w}_n)_n \subseteq \widetilde{W} \subseteq \widetilde{K}$ for $\widetilde{T}$. Then

$$\lim_n \|\widetilde{w}_n - x\| = diam\widetilde{K} = diam K = 1,$$

for any $x \in K$. Therefore

$$\sup\{\|\widetilde{w} - x\| : \ \widetilde{w} \in \widetilde{W}\} = diam K = 1,$$

for any $x \in K$. Since $0 \in K$, we get a contradiction. □

Corollary 2.38. *Every Banach space X with 1-unconditional basis $\{e_n\}_n$ has the weakly fixed point property.*

Proof. In such case, we have that $c = c_1 = c_2 = \mu = 1$. □

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